ASYMPTOTIC EFFICIENCIES FOR TWO-SAMPLE LINEAR RANK TESTS
IN THE PRESENCE OF RANDOM CENSORING

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CENSORED LINEAR RANK TESTS

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SUMMARY

Two samples, \( \{X_i, 1 \leq i \leq n_1\} \) and \( \{Y_i, 1 \leq i \leq n_2\} \) are assumed to be composed of iid random variables with survival functions \((1 - F_0)(1 - G)\) and \((1 - F_G)(1 - G)\), respectively, where \( G \) is the CDF of the "censoring times" and \( F_0 \) is the CDF of the "true lifetimes." A unified derivation of the Pitman efficiencies of two classes of linear rank statistics for censored samples is presented. The conditions under which the result holds do not require contiguous alternatives, since convergence to normality is shown to hold uniformly in \( \theta \) (for \( \theta \) in a compact set). The uniformity is obtained by considering the joint behavior of the censoring times and the true lifetimes. The statistic is constructed in terms of stochastic integrals with respect to the joint empirical distributions (one empirical distribution for each sample). The proof uses a result on central limit theorems for empirical measures (Dudley \( \text{Ann. Probab. 6:899-929, 1978} \)) and martingale central limit theorems. The results are applied to the translated exponential distributions, a non-contiguous family of alternatives.

Key Words and Phrases: Censored data, linear rank tests, Pitman efficiencies.

AMS Subject Classifications: Primary: 62G20; Secondary 62E20.

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I. INTRODUCTION AND NOTATION

Spurred by practical problems in clinical trials, many statisticians have proposed two-sample rank tests which can accommodate right censoring. This paper gives a unified derivation of the Pitman efficiency of a large class of linear rank tests for censored data under general, not necessarily contiguous, sequences of alternatives.

Efficiency calculations can be thought of as summarizing the behavior of sequences of statistics \( S_n \) and \( T_n \) under a sequence of alternatives indexed by \( \theta_n = \theta_o + cn^{-1/2} + o(n^{-1/2}) \). If, under the sequence of alternatives, \( n^{1/2} (T_n - \mu_n(\theta_n))/\sigma_n(\theta_n) \) is asymptotically standard normal and the functions \( \mu_n \) and \( \sigma_n \) are, respectively, uniformly continuously differentiable and uniformly continuous at \( \theta_o \), then the efficacy of the sequence \( T_n \) at \( \theta_o \) is given by the limit of \( (\mu'_n(\theta_o)/\sigma_n(\theta_o))^2 \). (Some refer to the square root of this quantity as the efficacy of \( T_n \).) The Pitman relative efficiency of the sequence \( T_n \), with respect to the sequence \( S_n \), is then the ratio of the efficacy of \( T_n \) to the efficacy of \( S_n \). The asymptotic normality of the statistics under the sequence of alternatives is
usually obtained from LeCam's third lemma, which implies that the joint asymptotic normality under $\theta^o$ of the sequence of statistics and the log likelihood ratio of $\theta_n$ to $\theta^o$, if the ratio of the limiting mean of the log likelihood ratio to the limiting variance is $-2$, is a sufficient condition for the asymptotic normality of the sequence of test statistics under the sequence of alternatives. (See Hájek and Sidák (1967), page 208.)

For noncontiguous alternatives (such as the translated exponentials of Example 4.3) the log likelihood ratio can fail to be asymptotically normal. (In Example 4.3, the log likelihood ratio converges to $-\infty$.) However (as in Example 4.3), the sequence of centered and scaled test statistics may nonetheless converge weakly to a normal distribution under every fixed alternative. Noether (1955) proved that in this case it suffices to show that the weak convergence of $n^{1/2}(T_n - \mu_n(\theta))/\sigma_n(\theta)$ is uniform in $\theta$ for $\theta$ in a neighborhood of $\theta^o$. Therefore Section 3 contains a proof of the uniformity in $\theta$ of the convergence in distribution of appropriately centered and scaled statistics.

The proof in Section 3 is based on the construction of a convenient version of the sequences of statistics. This construction is presented in Section 2. In the fourth section, the theorem of Section 3 is applied to particular tests and families of tests to verify that the efficiencies obtained from this approach are equivalent to known efficiencies and to obtain efficiencies not heretofore calculated. The concluding section discusses other applications of this approach.
This section concludes with a description of the details of the testing problem and the class of statistics to be considered.

Consider a parametric family of lifetime distribution \( F_\theta \) (\( \theta \) is a scalar) and suppose that \( Y_{11} \ldots Y_{1n_1} \) is a sample from \( F_0 \) and that \( Y_{21} \ldots Y_{2n_2} \) is an independent sample from \( F_0 \). The null hypothesis is that \( \theta' = 0 \). If the lifetimes are randomly censored, then \( C_{11}, \ldots, C_{1n_1}, C_{21}, \ldots, C_{2n_2} \) will be a sample, independent of the \( Y \)'s, from a censoring distribution \( H \). (The censoring distribution is assumed to be the same for both populations. See Section 5.) The observed lifetimes \( X_{ij} = \min(Y_{ij}, C_{ij}) \) and the indicators \( \delta_{ij} = I\{X_{ij} = Y_{ij}\} \) are the variables on which test statistics must be based. Define the random functions

\[
R_1(t) = \sum_{j=1}^{n_1} I\{X_{ij} > t\}, \quad \text{the number of individuals in the } i^{th} \text{ population "at risk at time } t \text{" and } D_1(t) = \sum_{j=1}^{n_1} I\{X_{ij} < t\} I\{\delta_{ij} = 1\}, \quad \\
\text{the number of "deaths" in the } i^{th} \text{ population before } t. \quad \text{Set } R_1(t) = R_1(t) + R_2(t), D_1(t) = D_1(t) + D_2(t) \text{ and } n = n_1 + n_2.
\]

The class of statistics

\[
T_n = \frac{1}{n} \int W_n(t) \left[ dD_1(t) - \frac{R_1(t)}{R_+(t)} \, dR_+(t) \right] \quad (1.1)
\]

where \( W_n(t) \) is a possibly random function determined by \( \{D_1(s), R_1(s), s \leq t, i = 1, 2\} \), includes many of the standard
censored data rank tests. This class of statistics includes two-sample forms of Prentice's linear rank tests, if a natural condition is imposed on the weights. (See Prentice and Marek (1979) expression (9)). If \( W_n^\ast(t) = j(R_+(t)/n) \), where \( j \) is a nonrandom weight function defined on \([0, 1]\), the class of tests studied by Tarone and Ware (1975) results.

If \( W_n^\ast(t) = j(t) \), a deterministic function, then \( T_n \) is the partial likelihood score statistic proposed in Cox (1972) for testing \( \beta = 0 \) in the model \( \lambda_\beta(t) = e^{\beta j(t)} \lambda_0(t) \). If \( W_n^\ast(t) = 1 - \hat{F}_0(t) \), the Kaplan-Meier estimator of \( F_0 \) obtained from the pooled sample, \( T_n \) is the generalization of the Wilcoxon test suggested by Peto and Peto (1972). The statistics (1.1) also form a subclass of the statistics mentioned in Aalen (1978) (see page 718).

It will be convenient below to define \( W_n(t) = nW_n^\ast(t)/R_+(t) \) in order to obtain a form of the statistic which is more clearly symmetric in the sample:

\[
T_n = \frac{1}{n} \int W_n(t) \left[ \frac{R_2(t)}{n} dD_1(t) - \frac{R_1(t)}{n} dD_2(t) \right]
\]

For technical reasons, it will be necessary to use a truncated statistic and to study

\[
T_n = \frac{1}{n} \int_0^T W_n(t) \left[ \frac{R_2(t)}{n} dD_1(t) - \frac{R_1(t)}{n} dD_2(t) \right].
\]
The notation below will be convenient in the following sections:

\[ \eta_0(B_1 \times B_2) = F_\theta(B_1) H(B_2) \quad \eta(t) = \eta_0(t) \]

\[ \lambda_\theta(s) = -f_\theta(s)/(1 - \Phi_\theta(s)) \quad \lambda(t) = \lambda_0(t) \]

\[ \Lambda_\theta(t) = \int_0^T \lambda_\theta(s) ds \quad \Lambda(t) = \Lambda_0(t) \]

\[ a_n(t) = \lambda(t) N_0(V_t) \quad a(t) = \lambda(t) \eta_0(V_t) \]

\[ A_n(t) = \int_0^T a_n(s) ds \quad A_n, \theta(t) = \int_0^T \lambda_\theta(s) N^{(2)}(\tau_\theta V_s) ds \]

\[ \alpha_\theta(t) = \eta(\tau_\theta V_t) \quad a(t) = \alpha_\theta(t) \]

\[ \beta_\theta(t) = \eta(\tau_\theta V_t) \quad \beta(t) = \beta_0(t) \]

\[ \mu_n(\theta) = \int_0^T \omega_n, \theta(t) \alpha_\theta(t) \alpha(t)(\lambda(t) - \lambda_\theta(t)) dt \]

\[ M_n(t) = N_n^{(1)}(B_t) - A_n(t) = N_n^{(1)}(B_t) - \int_0^T N_n(V_s) \lambda(s) ds \]

\[ Q_\theta(x) = \mathcal{P}_0^{-1} F_\theta(x) \]

\[ \tau_\theta(B) = \{(x,y) : (Q_\theta(x), y) \in B\} \]

\[ \gamma_{n, \theta}(t) = \frac{n_1}{n} \alpha(t) + \frac{n_2}{n} \alpha_\theta(t) \]

\[ \gamma_\theta(t) = p \alpha(t) + q \alpha_\theta(t) \]
II. THE CONSTRUCTION

This section describes the construction of a version of the test statistic based on two independent counting processes, neither of which depends on $\theta$. The version chosen is convenient in that it depends on $\theta$ in an explicit manner. The counting processes are constructed by considering the joint (unobservable) distribution of $Y_{ij}$ and $C_{ij}$.

For fixed $n_1$ and $n_2$, define counting processes $N_0$ and $N_0$ on the Lebesgue measurable sets of $\mathbb{R}^2$ by

$$N_0(B) = \sum_{j=1}^{n_1} I\{(Y_{1j}, C_{1j}) \in B\}$$

and

$$N_\theta(B) = \sum_{j=1}^{n_2} I\{(Y_{2j}, C_{2j}) \in B\}$$

The distributions of $N_0$ and $N_\theta$ can be specified by noting that $N_0(B)$ is binomial $(n_1, \eta_0(B))$ and $N_\theta(B)$ is binomial $(n_2, \eta_\theta(B))$, where $\eta_\theta(B_1 \times B_2) = F_\theta(B_1)H(B_2)$. (Whether $F_\theta$ and $H$ denote cumulative distribution functions or the associated probability measures will be apparent from context.) If $V_t = [t, \infty) \times [t, \infty)$ and $B_t = \{(x, y): x < t, y \geq x\}$, then $R_1(t) = N_0(V_t)$, $R_2(t) = N_\theta(V_t)$, $D_1(t) = N_0(B_t)$, and $D_2(t) = N_\theta(B_t)$. Note that $T_n$ depends on $N_0$ and $N_\theta$ only through the collection of sets $\{V_t, B_t, t \leq T\}$. (The
unsupplied diagrams may be helpful here.) Substituting in (1.1),

\[ T_n = \frac{1}{n} \int_0^T W_n(t) \left( \frac{N_0(V_t)}{n_1} \right) dN_0(B_t) - \frac{N_0(V_t)}{n_2} dN_0(B_t) \]

Write \( T_n = n_1 n_2 (T_{n1} - T_{n2}) \), where

\[ T_{n1} = \int_0^T W_n(t) \frac{N_0(V_t)}{n_2} \frac{dN_0(B_t)}{n_1} \]

and

\[ T_{n2} = \int_0^T W_n(t) \frac{N_0(V_t)}{n_1} \frac{dN_0(B_t)}{n_2} \]

Let \( \omega_{n, \theta}(t) \) be a nonrandom function and \( Z_{n\theta} \) be a random function.

(Conditions will be imposed on these functions in Theorem 3.1).

By adding and subtracting terms, the following expansion of

\( n^{1/2} T_{n1} \) can be obtained:
(2.1) \[ n^{1/2} T_{n,1} = n^{1/2} \int_0^T W_n(t) \frac{N_{\theta} V_t}{n_2} \frac{dN_0 B_t}{n_1} \]

\[ = \int_0^T \left[ n^{1/2} (W_n(t) - \omega_{n,\theta}(t)) - Z_{n,\theta}(t) \right] \frac{N_{\theta} V_t}{n_2} \frac{dN_0 B_t - a_n(t) dt}{n_1} \]

\[ + \int_0^T Z_{n,\theta}(t) \left[ \frac{N_{\theta} V_t}{n_2} - \eta_{\theta} V_t \right] \frac{dN_0 B_t - a_n(t) dt}{n_1} \]

\[ + \int_0^T \left[ \frac{N_{\theta} V_t}{n_2} - \eta_{\theta} V_t \right] \frac{a_n(t) dt}{n_1} \]

\[ + \int_0^T \omega_{n,\theta}(t) \frac{n^{1/2} \left( \frac{N_{\theta} V_t}{n_2} - \eta_{\theta} V_t \right)}{n_1} \frac{dN_0 B_t - a_n(t) dt}{n_1} \]

\[ + \int_0^T \omega_{n,\theta}(t) \frac{n^{1/2} \left( \frac{N_{\theta} V_t}{n_2} - \eta_{\theta} V_t \right)}{n_1} \frac{a_n(t) dt}{n_1} \]

\[ + \int_0^T \omega_{n,\theta}(t) \eta_{\theta} V_t \frac{n^{1/2} \left( \frac{dN_0 B_t - a_n(t) dt}{n_1} \right)}{n_1} \]

\[ + \int_0^T \omega_{n,\theta}(t) \eta_{\theta} V_t \frac{n^{1/2} a_n(t)}{n_1} \frac{\frac{a_n(t)}{n_1} - a(t)}{dt} \]

\[ + n^{1/2} \int_0^T \omega_{n,\theta}(t) \eta_{\theta} V_t \ a(t) dt \]

The last term is non-random, and will be denoted by \( n^{1/2} \int_0^T, 1(\theta) \).

The eighth term in the expansion can, if \( \omega_{n,\theta}(t) \eta_{\theta} V_t \) is absolutely continuous, can be rewritten as an integral with respect to a
deterministic measure:

\begin{equation}
\int \omega_n, \theta(t) \eta_0 V_t n^{1/2} \left( \frac{dN_0 B_t - a_n(t)}{n_{1/2}} \right) dt
\end{equation}

= \eta_0 V_t \omega_n, \theta(T) \left[ N_0 B_T - A_n(T) \right] n^{1/2} n_{1/2}^{-1}

+ \int_0^T (N_0 B_t - A_n(t)) d(\omega_n, \theta(t) \eta_0 V_t) n^{1/2} n_{1/2}^{-1}.

The quantity \( n^{1/2} T n_2 \) can be expanded similarly.

The proof in the next section uses versions of \( N_0 \) and \( N_0 \) that are tractable in \( n \) and in \( \theta \). To make \( N_0 \) vary tractably with \( \theta \), let \( Q_0(x) = F_0^{-1} F_\theta(x) \) and \( \tau_0(B) = \{ (x, y) : (Q_0(x), y) \in B \} \), so that \( Q_0 \) maps the \( p \)th quantile of \( F_\theta \) into the \( p \)th quantile of \( F_0 \) and \( \tau_0(B) \) is the pre-image of the set \( B \) under a mapping which transforms the horizontal coordinates (\( F_\theta \) lifetimes into \( F_0 \) lifetimes). Clearly \( \eta_0(B) = \eta_0(\tau_0(B)) \) and the set process \( N_0(\cdot) \) has the same distribution as the process \( N_0(\tau_0(\cdot)) \).

Therefore a version of the statistic can be constructed which depends on \( N_0 \) through the collection of sets \( \{ B_t, V_t, t \leq T \} \) and on \( N_0' \), an independent version of \( N_0 \), through the collection of sets \( \{ \tau_0 B_t, \tau_0 V_t, t \leq T \} \).

The process \( n_{1/2}^{1/2} \left( n_{1/2}^{-1} N_0(\cdot) - \eta(\cdot) \right) \) is, in the terminology of Dudley (1978), a normalized random empirical measure. To study the behavior of \( T_n \), it suffices to study such normalized measures on the collection of sets

\[ C = \{ \tau_0 V_t, \tau_0 B_t : \theta \in \Theta, t \leq T \} \]

If \( C \) is a Donsker class for \( \eta \), then there is a version of the
sequence of normalized random empirical measures and a Gaussian process \( g^{(1)} \) such that

\[
\lim_{n \to \infty} \sup_{C \in \mathcal{C}} \left| n^{1/2} \left( \frac{N_n^{(1)}(C)}{n} - \eta(C) \right) - g^{(1)}(C) \right| = 0. 
\]

(See the first paragraph of the proof of the converse of Dudley's Theorem 1.2.) The covariance structure of \( G \) is determined by \( \eta \) as described in the last paragraph of page 900 of Dudley (1978). Note that (2.3) forces

\[
\lim_{n \to \infty} \sup_{C \in \mathcal{C}} \left| n^{-1} N_n^{(1)}(C) - \eta(C) \right| = 0
\]

Heuristically, \( \mathcal{C} \) will be a Donsker class if it is not possible to select a member of \( \mathcal{C} \) so that the empirical measure gives that set either unusually high or unusually low probability. (Thus a class of sets containing all finite unions of discs of arbitrary center and arbitrary radius cannot be a Donsker set, since such a set \( \mathcal{C} \) can always be chosen to make \( \eta(C) \) arbitrarily small and \( N_n(C)/n \) one.) The following proposition summarizes some of Dudley's sufficient conditions for a class \( \mathcal{C} \) of subsets of \( \mathbb{R}^2 \) to be a Donsker class.
Proposition 2.1

Each of the following is sufficient for a class $C$ of subsets of $\mathbb{R}^2$ to be a Donsker class (and for (2.3) to hold):

(a) Every member of $C$ is a convex set and $F_\theta$ has bounded density with respect to Lebesgue measure.

(b) Every member of $C$ is of the form

\[(x,y) : h_1(y) \leq x \leq h_2/y, \quad (x,y) : h_3(x) \leq y \leq h_4(x),\]

where the functions $h_i$ have derivatives of order $a$ bounded in absolute value for every $a \leq 2$ or $h_i \equiv \infty$.

(The bounds need not be uniform across members of $C$).

See Dudley (1978), p. 918. Although these conditions are not the best possible, they generate much larger Donsker classes than are needed here. For the class of sets under consideration here, the Donsker class property can be determined from the properties of $Q_\theta$.

Corollary 2.1

If $Q_\theta$ satisfies either of the two conditions below, then $C = \{\tau_\theta F_t, \tau_\theta V_t ; t \leq T, \theta \in \Theta\}$ is a Donsker class.

(a) $Q_\theta(x) = F_0^{-1}F_\theta(x)$ is a convex function of $x$ for every $\theta$ and $F_\theta$ has a bounded density.

(b) $F_\theta$ is twice differentiable, $\sup_{0 \leq t \leq T} f_\theta'(t) < \infty$, and $f_\theta$ is bounded away from zero on $[0, F^{-1}_\theta(T)]$ for every $\theta$. 
The first condition will be met by any location, scale or one parameter location-scale family with a bounded density. The second condition can be examined for other families.

**Proof of Corollary**

Since \( \tau_{\theta} B_{t} = \{(x,y) : Q_{\theta}(x) < y]\cap\{(x,y) : 0 \leq x < Q_{\theta}^{-1}(t)\} \)

\( \tau_{\theta} V_{t} = \{(x,y) : x \geq Q_{\theta}^{-1}(t)\} \cap \{(x,y) : y \geq t\} \), and since \( Q_{\theta}^{-1}(t) \) and \( t \) are trivially convex, differentiable functions of \( y \), the applicability of Proposition 2.1 depends only on the properties of \( Q_{\theta}(x) \). If \( Q_{\theta}(x) \) is convex, then (a) of Proposition 2.1 holds.

If (b) of Corollary 2.1 holds, then

\[
\frac{d^2}{dx^2} Q_{\theta}(x) = \frac{f'_{\theta}(x)}{f(F^{-1}(F_{\theta}(x)))} - \frac{f^2_{\theta}(x)f'(F^{-1}(F_{\theta}(x)))}{f^3(F^{-1}(F_{\theta}(x)))}
\]

is bounded and (b) of the proposition holds.
Let \( N^{(2)}_n \) and \( G^{(2)} \) be independent copies of \( N^{(1)}_n \) and \( G^{(1)} \) that also satisfy (2.3). The pair of processes \( N^{(1)}_{n_1}(\cdot) \) and \( N^{(2)}_{n_2}(\tau_\theta(\cdot)) \) (to be denoted \( N^{(1)}_n(\cdot) \) and \( N^{(2)}_n(\tau_\theta(\cdot)) \) hereafter) has the same distribution as the pair of processes \( N_0 \) and \( N_\theta \). Therefore \( N_0 \) can be replaced by \( N^{(1)}_n \) and \( N_\theta \) and \( N^{(2)}_n(\tau_\theta) \) in the statistics above without affecting any distributional properties. In particular, the equations (2.1) and (2.2) imply that the following quantity has the correct distribution:

\[
n^{1/2}(T_{n_1} - \mu_{n_1}(\theta)) = \]

(2.5.1) \[
\int_0^T n^{1/2}(W_{n, \theta}(t) - \omega_{n, \theta}(t)) - Z_{n, \theta}(t) \frac{N^{(2)}_n(\tau_\theta V_t)}{n_2} \frac{dN^{(1)}_n(B_t)}{n_1} + \int_0^T Z_{n, \theta}(t)[\frac{N^{(2)}_n(\tau_\theta V_t)}{n_2} - \eta(\tau_\theta V_t)] \frac{[dN^{(1)}_n(B_t) - a_n(t)dt]}{n_1} \]

(2.5.2) \[
+ \int_0^T Z_{n, \theta}(t)[\frac{N^{(2)}_n(\tau_\theta V_t)}{n_2} - \eta(\tau_\theta V_t)] \frac{a_n(t)dt}{n_1} \]

(2.5.3) \[
+ \int_0^T Z_{n, \theta}(t)\eta_{\tau_\theta V_t} \frac{[dN^{(1)}_n(B_t) - a_n(t)dt]}{n_1} \]

(2.5.4) \[
+ \int_0^T Z_{n, \theta}(t)\eta_{\tau_\theta V_t} \frac{a_n(t)dt}{n_1} \]

(2.5.5) \[
+ \int_0^T \omega_{n, \theta}(t) \left[ n^{1/2} \frac{N^{(2)}_n(\tau_\theta V_t)}{n_2} - \eta(\tau_\theta V_t) \right] \frac{dN^{(1)}_n(B_t) - a_n(t)dt}{n_1} \]
\begin{align*}
(2.5.7) & \quad + \int_0^T \omega_{n, \theta}(t)[n^{1/2} \left( \frac{N^{(2)}(\tau_\theta V_t)}{n_2} - \eta(\tau_\theta V_t) \right)] \frac{a_n(t)dt}{n_1} \\
(2.5.8) & \quad + n^{1/2} \eta(\tau_\theta V_T) \omega_{n, \theta}(T) \left[ \frac{N^{(1)}(B_n)}{n_1} - \int_0^T \frac{a_n(s)ds}{n_1} \right] \\
(2.5.9) & \quad + - \int_0^T n^{1/2} \left( \frac{N^{(1)}(B_t)}{n_1} - \int_0^t \frac{a_n(s)ds}{n_1} \right) \omega_{n, \theta}(t) \eta(\tau_\theta V_t) \\
(2.5.10) & \quad + \int_0^T \omega_{n, \theta}(t) \eta(\tau_\theta V_t) n^{1/2} \left( \frac{a_n(t)}{n_1} - a(t) \right) dt 
\end{align*}

The corresponding expression for $n^{1/2}(T_{n,2} - \mu_{n,2}(\theta))$ is given in the appendix.
III. PROOF OF UNIFORM ASYMPTOTIC NORMALITY

This section contains the proof of the uniform asymptotic normality of the appropriately centered and scaled form of $T_n$ under general assumptions about the weights and the distributions:

**Assumptions** The set $\Theta$ is a neighborhood of 0 and $T$ is fixed scalar.

The processes $W_{n,\theta}$, $Z_{n,\theta}$, the Gaussian process $Z_\theta$ and the deterministic functions $\omega_{n,\theta}$ and $\omega_\theta$ satisfy:

A1. $\lim_{n \to \infty} \sup_{\theta \in \Theta} \sup_{t} \left| n^{1/2} (W_{n,\theta}(t) - \omega_{n,\theta}(t)) - Z_{n,\theta}(t) \right| = 0$

A2. $P\left( \sup_{\theta \in \Theta} \sup_{0 \leq t \leq T} | Z_{n,\theta}(t) | < \infty \right) = 1$ for every $n$.

A3. $\lim_{n \to \infty} \sup_{\theta \in \Theta} \sup_{0 \leq t \leq T} | Z_{n,\theta}(t) - Z_\theta(t) | = 0$

A4. The distributions $F_\theta$ are absolutely continuous and satisfy $\sup_{0 \leq t \leq T} \sup_{\theta \in \Theta} \lambda_\theta(t) < \infty$

A5. The functions $\omega_\theta(t)\alpha_\theta(t)$ possess a Radon-Nikodym derivative with respect to Lebesgue measure for every $\theta$ in $\Theta$, and the derivatives are bounded uniformly in $t$ and in $\Theta$.  

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A6. The Radon-Nikodym derivatives of $\omega_n, \theta(t) \alpha(t)$ with respect to Lebesgue measure exist, are bounded uniformly in $n$, $\theta$, and $t$ and converge (as $n$ diverges) to the Radon-Nikodym derivative of $\omega_0(t) \alpha_0(t)$ (for $t, \theta$) fixed in $[0, T] \times \Theta$.

A7. Either a) or b) of Corollary 2.1 holds.

A8. $\omega_\theta(0) = 1$ for all $\theta$.

A9. $\lim_{n \to \infty} (n_1/n) = p = 1 - q \in (0, 1)$.

In the case that $W_n, \theta(t)$ is deterministic, assumptions 1, 2, and 3 are superfluous.

**Theorem 3.1**

Under the assumptions A1 through A8, $n^{1/2} (T_n - \mu_n (\theta))$ converges in distribution to a mean zero Gaussian random variable $L(\theta)$ uniformly in $\theta$. The variance of $L(\theta)$ is a continuous function of $\theta$ and the variance of $L(0)$ is given by (3.5) through (3.8) below.

Since to show the uniform convergence in distribution of $n^{1/2} (T_n - \mu(\theta))$, it suffices to show the uniform pointwise convergence of the version constructed in Section 2, in the rest of this section $T_n (T_{n1}, T_{n2})$ will denote the version of $T_n (T_{n1}, T_{n2})$ based on $N_n (1)$ and $N_n (2)$. 
Proposition 3.1

\[ \lim_{n \to \infty} n^{1/2}(T_{n1} - \mu_{n,1}(\theta)) = L_1(\theta) \] uniformly in \( \theta \)

where

\[ L_1(\theta) = \int_0^T Z_\theta(t) \alpha(t) \alpha_\theta(t) \lambda(t) dt + q^{-1/2} \int_0^T \psi_\theta(t) \alpha(t) G^{(2)}(\theta) \tau_\theta V_\ell \lambda(t) dt \]

\[ + p^{-1/2} \left[ \omega_\theta(T) \alpha_\theta(T) G^{(1)}(B_\ell) - \int_0^T G^{(1)}(B_\ell) d(\omega_\theta(t) \alpha_\theta(t)) \right]. \]

Proof of Proposition

The proposition will be proven by showing that five of the terms of the expansion (2.5) converge to zero uniformly in \( \theta \). Assumption A7 implies the convergence (2.3), which will be used to obtain limiting random variables.

The assumption A1 implies immediately that (2.5.1) converges uniformly to 0, since

\[ |(2.5.1)| \leq \sup_{0 \leq t \leq T} \left| n^{1/2}(W_{n,\theta}(t) - \omega_n \theta(t)) - Z_{n,\theta}(t) \right|. \]

The inequality

\[ |(2.5.3)| \leq \sup_{0 \leq t \leq T} |Z_{n,\theta}(t)| \]

\[ + \sup_{\theta \in \Theta, t \leq T} \left| \frac{N_n^{(2)}(\tau_0 V_\ell)}{n} - \eta(\tau_0 V_\ell) \right| A(T) \]

shows that the assumptions A2, A3 and (2.4) force (2.5.3) to converge uniformly to zero. Lemma 3.1 below contains the proof that (2.5.2), (2.5.4) and (2.5.6) converge uniformly to zero.
The uniform convergence of the sum of the remaining terms to $L_1(\theta)$ follows from (2.3). Thus the assumptions A3, A4 and (2.4) imply that at every point in the underlying sample space (2.5.5) converges uniformly in $\theta$ to

$$f_0^t z_\theta(t)\eta(V_t)\eta(\tau_0 V_t)\lambda(t)dt.$$ 

Similarly, A4, A5, A6, and (2.3) imply that (2.5.7) converges uniformly in $\theta$ to

$$q^{-1/2}f_0^T \omega_\theta(t)G^{(2)}(t_0 V_t)\lambda(t)\eta(V_t)dt$$

and that (2.5.10) converges uniformly in $\theta$ to

(3.1) \quad p^{-1/2}f_0^T \omega_\theta(t)G^{(1)}(V_t)\eta(\tau_0 V_t)\lambda(t)dt.

The two remaining terms follow from the convergence (uniformly in $t$ and $\theta$) implied by (2.3) and A4 of $\eta^{-1/2}_n (N^{(1)}_n (V_t) - A_n(t))$ to

$$G^{(1)}(B_t) - f_0^T G^{(1)}(V_s)\lambda(s)ds$$

The assumptions A5, A6 and A8 imply that the sum of (2.5.8) and (2.5.9) converges uniformly to

$$p^{-1/2}[\omega_\theta(t)\eta(\tau_0 V_t)G^{(1)}(B_t) - f_0^T G^{(1)}(B_t)d(\omega_\theta(t)\eta(\tau_0 V_t))]$$

(3.2) \quad - f_0^T \omega_\theta(t)\eta(\tau_0 V_t)\lambda(t)G^{(1)}(V_t)dt].$$

Proposition 3.1 now follows. (The last term of (3.1) cancels (3.2).)
Proposition 3.2

The version of $\tau_{n2}$ constructed in Section 2 satisfies

$$\lim_{n \to \infty} n^{1/2}(\tau_{n2} - \mu_{n2}(\theta)) = L_{2}(\theta)$$

uniformly in $\theta$, where

$$L_{2}(\theta) = \int_{\delta}^{T} Z_{\theta}(t)\alpha(t)\lambda_{\theta}(t)dt$$

$$+ p^{-1/2} \int_{\delta}^{T} \omega_{\theta}(t)\alpha_{\theta}(t)\lambda_{\theta}(t)G^{(1)}(\nu_{e})dt$$

$$+ q^{-1/2} \int_{\delta}^{T} \omega_{\theta}(t)\alpha(t)G^{(2)}(\nu_{e}B_{e}) - \int_{\delta}^{T} G^{(2)}(\nu_{e}B_{e})d(\omega_{\theta}(t)\alpha(t)).$$

The proof of Proposition 3.2 follows that of Proposition 3.1 and is omitted.

Lemma 3.1

Each of (2.5.2), (2.5.4) and (2.5.6) converges to zero uniformly in $\theta \in \Theta$.

These three terms involve integration with respect to the process $M_{n}(t)$. The proof that these stochastic integrals converge to zero uniformly in $\theta$ will use the fact that a filtration $F_{n}$ can be defined with the property that $M_{n}(t)$ is a martingale with respect to this filtration. Such a filtration is $F_{n}$

$$= \{F_{n,t} : t \geq 0\},$$

where

$$F_{n,t} = \sigma\{N_{n}^{(1)}(u,v), N_{n}^{(2)}(x,y), u \wedge v \leq t, x \geq 0, y \geq 0\}.$$  

(See Example 1 of Aalen (1978).) The processes $R_{1}(t)$, $R_{2}(t)$, $D_{1}(t -)$, and $D_{2}(t -)$ are left-continuous and adapted to $F_{n}$.
The theory of integration with respect to martingales can be used to bound the variances of (2.5.2), (2.5.4) and (2.5.6). The facts needed are stated here. For a more complete discussion, see Aalen (1978), Dellacherie (1980) or the references in these articles.

If $M_t$ is a martingale with respect to a filtration $\mathcal{F} = \{\mathcal{F}_t, \ t \geq 0\}$, and $M_0 = 0$ and $E[M^2(t)] < \infty$ for $t \geq 0$, there is a non-decreasing process $A_M(t)$ (often denoted $\langle M, M \rangle_t$ and called the variance process of $M$ or the compensator of $M$) such that $M^2(t) - A_M(t)$ is a martingale. As a consequence, $E[M^2(t)] = E[A_M(t)]$, or $A_M$ is an unbiased estimator of the variance of $M$. If $X$ is a predictable process such that $E[\int_0^TX^2(s)dM(s)] < \infty$, the stochastic integral $Z(t) = \int_0^TX(s)dM(s)$ is also a martingale. (A sufficient condition for a process to be a predictable process is that the process be adapted and left-continuous.) Furthermore, $A_Z$, the compensator of $Z$, is given by $A_Z(t) = \int_0^TX^2(s)dA_M(s)$.

Aalen (1978) gives a proof of the fact that $A_n$ is the compensator of $M_n$. Therefore the compensator of (2.5.2) is given by

$$\int Z_{n, \theta}^2(t) \left[ \frac{N^{2}(\tau_{\theta} V_t)}{n_2} - \eta(\tau_{\theta} V_t) \right]^2 \frac{dA_n(t)}{n_2^2}$$

$$\leq \frac{Z}{n_1} \Lambda(T).$$

The convergence to zero of (2.5.2) uniformly in $\theta$ in a consequence of Lemma 2 below. (The dependence of the filtration on $\theta$ is used
in the proof of Proposition 2). The proofs for (2.5.4) and 
(2.5.6) are similar.

Lemma 3.2

Assume that for each \( n \) and \( \theta \), \( X_{n,\theta} \) is a square-integrable 
martingale with respect to the filtration \( F_{n,\theta} \). Also assume that 
\( X_{n,\theta}(0) = 0 \) and that \( A_{n,\theta} \) is the compensator of \( X_{n,\theta} \). If, for 
every positive \( \varepsilon \),

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta} P\{A_{n,\theta}(T) > \varepsilon\} = 0
\]

Then \( \lim_{n \to \infty} \sup_{\theta \in \Theta} P\{|X_{n,\theta}(T)| > \varepsilon\} = 0 \quad \forall \varepsilon > 0 \).

Proof. The notation in the statement and proof in this lemma is 
independent of the rest of the paper.

Define a family of stopping times by

\[
\tau_{n,\theta} := \inf \{ t : A_{n,\theta}(t) > 1 \}
\]

Next define the stopped processes \( X_{n,\theta}^*(t) := X_{n,\theta}(t \wedge \tau_{n,\theta}) \) and 
\( A_{n,\theta}^*(t) := A_{n,\theta}(t \wedge \tau_{n,\theta}) \). The compensator of \( X_{n,\theta}^* \) is \( A_{n,\theta}^* \). These 
definitions imply that

\[
(3.3) \quad P\{ \sup_{t \leq T} |X_{n,\theta}(t)| > \varepsilon \} \leq P\{ \sup_{t \leq T} |X_{n,\theta}^*(t)| > \varepsilon \} + P\{\tau_{n,\theta} < T\}.
\]

Applying Chebychev's inequality and then the martingale inequality,
the first term of (3.3) is bounded by
\[ 4 \sup_{t \leq T} \mathbb{E}[(X_{n,\theta}^*(t))^2] \leq 4 \varepsilon^{-2} \sup_{t \leq T} A_{n,\theta}^*(t). \]

Since compensator processes are non-decreasing, (3.3) is bounded above by
\[ 4 \varepsilon^{-2} \mathbb{E}[A_{n,\theta}^*(T)] + \mathbb{P}(\tau_{n,\theta} \leq T) \]
\[ \leq 4 \varepsilon^{-2} (\varepsilon^3 + \mathbb{P}(A_{n,\theta}(T) > \varepsilon^3)) + \mathbb{P}(A_{n,\theta}(T) > 1). \]

The hypotheses of the lemma then imply that this bound converges to \( 4 \varepsilon \) uniformly in \( \theta \).

**Proof of Theorem**

Propositions 1 and 2 imply that \( n^{1/2}(T_n - \mu(\theta)) \) converges uniformly in \( \theta \) to
\[
(3.4) \quad L(\theta) = \int_0^T \omega_\theta(t) \alpha(t) \alpha_\theta(t) \lambda(t) dt - \int_0^T \omega_\theta(t) \alpha(t) \alpha_\theta(t) \lambda_\theta(t) dt
\]
\[ + q^{-1/2} \left[ \int_0^T \omega_\theta(t) \alpha(t) G^{(2)}(\tau_\theta \nu_t) \lambda(t) dt - \omega_\theta(T) \alpha(T) G^{(2)}(\tau_\theta B_T) 
\]
\[ + \int_0^T G^{(2)}(\tau_\theta B_T) d(\omega_\theta(t) \alpha(t)) \right]
\[ + p^{1/2} \left[ \omega_\theta(T) \alpha_\theta(T) G^{(1)}(B_T) - \int_0^T G^{(1)} B_T d(\omega_\theta(t) \alpha_\theta(t)) \right.
\]
\[ - \int_0^T \omega_\theta(t) \alpha_\theta(t) G^{(1)}(\nu_t) \lambda_\theta(t) dt \right]. \]

The boundedness of \( \omega_\theta, \lambda_\theta \) and the Radon-Nikodym derivative of \( \omega_\theta(t) \alpha_\theta(t) \) imply that \( L(\theta) \) is a continuous linear functional (from \( C[0,1] \)) of the three Gaussian processes \( Z_\theta, G^{(1)} \) and \( G^{(2)} \). Therefore \( L(\theta) \) will have a Gaussian distribution with finite variance. Since, for each point in the underlying sample space,
$L(\theta)$ is a continuous function of $\theta$, the variance of $L(\theta)(\sigma^2(\theta))$ converges to the variance of $L(0)$, $\sigma^2(0)$. When $\theta = 0$, the first two terms of (3.4) cancel each other and $L(0)$ is the weighted sum of two independent Gaussian random variables, each of the form

$$\int_0^{3T} q(t)X(t)dt - q^*X(T),$$

where

$$q^* = \omega_0(T)\alpha(T)$$

and

$$q(t) = \frac{d(\omega_0(t)\alpha(t))}{dt}I_{0 \leq t \leq T} + \omega_0(t-2T)\alpha(t-2T)\lambda(t-2T)I_{2T \leq t \leq 3T}$$

and

$$X(t) = G(B_t)I_{0 \leq t \leq T} + G(V_{t-2T})I_{2T \leq t \leq 3T}$$

The covariance function of $X$ is $K(s, t)$, where for $s \leq t$,

$$K(s, t) = \beta(s)(1 - \beta(t))I_{0 \leq s \leq t \leq T} + [(\beta(s) - \beta(t-2T)) + \alpha(t-2T)\beta(s)]I_{0 \leq s \leq T}I_{2T \leq t \leq 3T} + \alpha(t-2T)(1 - \alpha(s-2T))I_{2T \leq s \leq t \leq 3T}.$$

Therefore the variance of $L(0)$ is

$$\sigma^2(0) = (pq)^{-1}\left[\int_0^{3T} \int_0^{3T} q(s)K(s, t)q(t)dtsd - 2q^* \int_0^{3T} q(s)K(s, T)ds + q^*K(T, T)q^*\right].$$
IV. PITMAN EFFICIENCIES

In this section, the Pitman efficiencies of two classes of test statistics, those proposed by Peto and Peto (1972) and by Tarone and Ware (1975) are presented. The calculations for generalized Savage (Example 4.1) and generalized Wilcoxon tests (Example 4.2) are presented for general alternatives and for translated exponential alternatives (Example 4.3).

The calculation of Pitman efficiencies does not use the full force of Theorem 3.1: the processes $Z_{n,\theta}$ are needed to show uniform convergence to random variables depending on a limiting process $Z_\theta$. However, once the existence of these processes is checked, their properties are not needed in the evaluation of Pitman efficiencies. In particular, the covariance of $Z_\theta$ need not be computed. This covariance is needed for computation of $\sigma^2(\theta)$, but since $Z_\theta$ appears only in an integral where it is multiplied by $\lambda_0 - \lambda_0$, the covariance of $Z_0$ does not enter the calculation of $\sigma^2(0)$. The functions $\omega_\theta$, $\alpha$ and $\beta$ determine the variance $\sigma^2(0)$; the functions $\omega_\theta$, $\alpha$ and $\beta$ determine $\mu(\theta)$.

In this section, the efficacies of two classes of linear rank tests will be described. The two classes of statistics are given by (1.1) and the following weight functions:
S1: $W^*_n(t) = \frac{j(1 - \hat{F}_n(t))}{(R_+^*(t))/n}$, where \( \hat{F} \) is the Kaplan-Meier estimator based on the pooled sample 
\( \{X_{ij}, j \leq n_i, i = 1, 2\} \), \( j(1) = 1 \); \( j \) is twice continuously differentiable on \([0, \nu^*]\), where 
\( \nu^* = \sup_{\theta \in \Theta} \frac{F'_\theta(T)}{2} \)

S2: $\tilde{W}^*_n(t) = J(R_+^*(t))/n$, \( J(1) = 1 \), and \( J \) is twice differentiable on \([0, \nu^*]\).

The statistics satisfying S1 are the type of test described by Peto and Peto (1972). The statistics satisfying S2 are the class described by Tarone and Ware (1975). The generalized Wilcoxon test of Gehan (1965) is an S2 test; the generalized Wilcoxon test of Peto and Peto (1972) is (disregarding their use of Altshuler's (Nelson's) cumulative hazard estimator in place of the Kaplan-Meier survival function estimator) an S1 test. The logrank test is a member of both classes.

The following assumptions will be used in this section:

B1. The distributions \( F_\theta \) are absolutely continuous and satisfy \( \sup_{\theta \in \Theta} \sup_{0 \leq t \leq T} \lambda_\theta(t) < \infty \).

B2. \( \sup_{0 \leq t \leq T} (F_\theta(t) - F_0(t)) = O(\theta) \)

B3. Either a) or b) of Corollary 2.1 holds

B4. \( \theta_n = k^{-1/2} \)

B5. \( \lim_{n \to \infty} n_1/n = p, \, 0 < p < 1. \)
Note that the assumptions given on page 25 do not force $F_\theta$ to be a contiguous family of alternatives.

The theoretical content of this section is the following corollary to Theorem 3.1:

**Corollary 4.1**

Under the assumptions B1 through B5 and S1 or S2, 

$$n^{1/2}(\frac{\hat{T}_n}{n} - \mu(\hat{\theta}_n)) \xrightarrow{d} N(0, \sigma^2),$$

where $\mu(\hat{\theta})$ is given in the appropriate column of Table 4.1 and $\sigma^2$ is given in Lemma 4.2.

If $\mu$ is differentiable at 0, the efficacy of $T_n$ under the family $F_\theta$ is given by $(\mu'(0))^2/\sigma^2$.

**Proof of Corollary**

The corollary will follow, once the assumptions of Theorem 3.1 are checked and the covariance calculated. Assumption B1 is A4, B5 is A9, B4 is A7, and both S1 and S2 include A8. The assumptions A1 through A3 involve $Z_{n,\theta}$ and $Z_\theta$; assumptions A1, A5, and A6 involve $\omega_{n,\theta}$ and $\omega_\theta$. Table 4.1 gives the appropriate functions and processes for the two types of statistics. Assumptions A2, A3, A5 and A6 are consequences of the choices of Table 4.1 and the differentiability conditions obtained in S1 and S2. The remaining assumption follows from the differentiability conditions, Lemma 4.1 below and Lemma 1 of Breslow and Crowley (1974). The continuity of the second derivative of $j$ in S1 ensures that the transformation from the process of Lemma 4.1 is sufficiently smooth.
<table>
<thead>
<tr>
<th>\textbf{S1 Tests}</th>
<th>\textbf{S2 Tests}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_n^*(t) = \frac{j(1 - \hat{F}<em>n(t))}{R</em>+(t)/n}$</td>
<td>$J\left(\frac{R_+(t)}{n}\right) = \frac{j(R_+(t)/n)}{R_+(t)/n}$</td>
</tr>
<tr>
<td>$\omega_{n,\theta}(t) = \frac{j(1 - [n_1 F(t) + n_2 F_\theta(t)]/n)}{\gamma_{n,\theta}(t)}$</td>
<td>$J(\gamma_{n,\theta}(t))$</td>
</tr>
<tr>
<td>$\omega_\theta(t) = \frac{j(1 - [pF(t) + qF_\theta(t)])}{\gamma_\theta(t)}$</td>
<td>$J(\gamma_\theta(t))$</td>
</tr>
<tr>
<td>$\omega_\alpha(t) = \frac{j(1 - F(t))}{\alpha(t)}$</td>
<td>$\frac{j(\alpha(t))}{\alpha(t)} = J(\alpha(t))$</td>
</tr>
<tr>
<td>$Z_{n,\theta}(t) = n^{1/2} \left{ \frac{j'(1 - [n_1 F(t) + n_2 F_\theta(t)]/n)}{\gamma_{n,\theta}(t)} \left(\frac{R_+(t)}{n} - \gamma_{n,\theta}(t)\right) \right}$</td>
<td>$J'(\gamma_{n,\theta}(t)) n^{1/2} \left[ \frac{R_+(t)}{n} - \gamma_{n,\theta}(t) \right]$</td>
</tr>
</tbody>
</table>

(continued on page 28)
TABLE 4.1
CONSTANTS AND PROCESSES FOR THE TESTS DEFINED BY S1 OR S2
continued

\begin{align*}
Z_\theta(t) &= \frac{j'(1 - [pF(t) + qF_\theta(t)])}{\gamma_\theta(t)} \left[ p^{1/2} G_1(v_t) + q^{1/2} G_2(\tau_\theta v_t) \right] \quad \text{S1 Tests} \\
& \quad - j (1 - [pF(t) + qF_\theta(t)]) \frac{X_\theta(t)}{\gamma_\theta^2(t)} \quad \text{S2 Tests}
\end{align*}

\begin{align*}
\mu_n(\theta) &= \frac{\int_0^T j(1 - [n_1 F(t) + n_2 F_\theta(t)]/n)}{\gamma_{n,\theta}(t)} \alpha(t) \alpha_\theta(t) \left( \lambda(t) - \lambda_\theta(t) \right) dt \\
\mu(\theta) &= \frac{\int_0^T j(1 - [pF(t) + qF_\theta(t)])/Y_\theta(t)}{\gamma_\theta(t)} \left( \lambda(t) - \lambda_\theta(t) \right) dt
\end{align*}
Lemma 4.1

Let \( \hat{\Lambda}_n(t) = \int_0^t \frac{dP_+(s)}{R_+(s)} \). Then the sequence of pairs of processes

\[
\left( \begin{array}{c}
R_+(t)/n - \gamma_n, \theta(t) \\
\hat{\Lambda}_n(t) + \ln(1 - [n_1 F(t) + n_2 F_\theta(t)]/n)
\end{array} \right)
\]

converges weakly in \( D^2[0, T] \) and uniformly in \( \theta \) to a bivariate Gaussian process having the same distribution as

\[
(p^{1/2} G^{(1)}(v_t) + q^{1/2} G^{(2)}(\tau_\theta v_t),
\]

\[
p^{1/2} \int_0^t G^{(1)}(B_s) \frac{d\gamma_0(s)}{\gamma_0^2(s)} - \int_0^t G^{(1)}(V_s) \frac{\lambda(s)}{\gamma_0(s)} ds + \frac{G^{(1)}(B_t)}{\gamma_0(t)}
\]

\[
+ q^{1/2} \int_0^t G^{(2)}(\tau_\theta B_s) \frac{d\gamma_0(s)}{\gamma_0^2(s)} - \int_0^t G^{(2)}(\tau_\theta V_s) \frac{\lambda(s)}{\gamma_0(s)} ds + \frac{G^{(2)}(\tau_\theta B_t)}{\gamma_0(t)}
\]

\[
+ pq \int_0^t \frac{1}{\gamma_0^2(s)} \left[ \alpha(s) - \alpha(s) \frac{G^{(1)}(V_s)}{\rho^{1/2}} - \alpha(s) \frac{G^{(2)}(\tau_\theta V_s)}{\rho^{1/2}} \right] d\lambda(s) - \lambda_0(s) ds)
\]

Proof

It suffices to check that the versions constructed in Section 2 converge pointwise to limiting Gaussian processes and that this convergence is uniform in \( \theta \). Since the versions converge pointwise, the two sequences of processes can be examined separately.

The convergence of the first term is a consequence of (2.3), since \( n^{1/2} (R_+(t)/n - \gamma_n, \theta(t)) = (n_1/n)^{1/2} \left[ N^{(1)}(V_t)/n_1 - \eta(V_t) \right] + (n_2/n)^{1/2} \left[ N^{(2)}(\tau_\theta V_t)/n_2 - \eta(\tau_\theta V_t) \right] \). To see that the convergence
of the second term is also a consequence of (2.3),
\[ n^{1/2} \left( \hat{\Lambda}_n(t) - \left[ -\ln(1 - (pF(t) + qF_0(t))) \right] \right) \] is rewritten (using the assumption of equal censoring to rewrite
\[ -\ln(1 - (pF(t) + qF_0(t))) :\]

\[ n^{1/2} \left( \hat{\Lambda}_n(t) - \int_0^t \frac{p\mu(s)\lambda(s) + q\alpha_0(s)\lambda_0(s)}{\gamma_0(s)} \, ds \right) = \]

\[ \int_0^t \left( \frac{n}{N(1)(V_s) + N(2)(\tau_0 V_s)} - \frac{1}{\gamma_0(s)} \right) \frac{d[N(1)B_s - A_n(s)]}{n} \]

\[ + \left( \frac{n_1}{n} \right)^{1/2} \int_0^t \left( \frac{n}{N(1)(V_s) + N(2)(\tau_0 V_s)} - \frac{n^2}{n} \right) \frac{d\gamma_0(s)}{\gamma_0^2(s)} \]

\[ + \left( \frac{n_1}{n} \right)^{1/2} \int_0^t \frac{n_1}{n} \frac{N(1)(B_t)}{n} \frac{d\gamma_0(s)}{\gamma_0^2(s)} \frac{\lambda_1(s)ds}{\gamma_0(s)} \]

\[ + \int_0^t \left( \frac{n}{N(1)(V_s) + N(2)(\tau_0 V_s)} - \frac{1}{\gamma_0(s)} \right) \frac{d[N(2)(\tau_0 B_s) - A_n,\theta(s)]}{n} \]

\[ + \left( \frac{n_2}{n} \right)^{1/2} \int_0^t \left( \frac{n}{N(2)(\tau_0 B_s)} - \frac{n^2}{n} \right) \frac{d\gamma_0(s)}{\gamma_0^2(s)} \frac{\lambda_2(s)ds}{\gamma_0(s)} \]

\[ + \left( \frac{n_2}{n} \right)^{1/2} \frac{1}{\gamma_0(\tau)} \frac{n_2}{n} \frac{N(2)(\tau_0 B_t)}{n_2} - \eta(\tau_0 B_t)] \]
\[- \left(\frac{\overline{X}_2}{n}\right)^{1/2} \int_0^t \frac{1}{n} n_2^{1/2} \left[ \frac{N(2)(\tau \theta V_s)}{n_2} - \eta(\tau \theta V_s) \right] \frac{\lambda_0(s)}{\gamma_\theta(s)} ds \]

\[+ \int_0^t n^{1/2} \left[ \frac{N(1)\gamma_\theta(s)}{N(1)(V_s) + N(2)(\tau \theta V_s)} - \frac{\rho(s)}{\gamma_\theta(s)} \right] (\lambda(s) - \lambda_0(s)) ds \]

\[+ \frac{n_1}{n^{1/2}} \left[ \int_0^t \frac{\beta(s)}{\gamma_\theta(s)} ds + \frac{\beta(t)}{\gamma_\theta(t)} - \int_0^t \frac{d\beta(s)}{\gamma_\theta(s)} \right] \]

\[+ \frac{n_2}{n^{1/2}} \left[ \int_0^t \frac{\beta_0(s)}{\gamma_\theta(s)} ds + \frac{\beta_0(t)}{\gamma_\theta(t)} - \int_0^t \frac{d\beta_0(s)}{\gamma_\theta(s)} \right].\]

The first and fifth terms are integrals with respect to martingales; their variance processes are bounded by

\[\sup_{0 \leq t \leq T} \left[ n^{1/2} (n/(N(1)(V_s) + N(2)(\tau \theta V_s)))^2 \Lambda(T)/n \right] \]

and

\[\sup_{0 \leq t \leq T} \left[ n^{1/2} (n/(N(1)(V_s) + N(2)(\tau \theta V_s)))^2 \Lambda_\theta(T)/n, \right] \]

respectively.

Lemma 3.2 therefore implies these terms converge to zero uniformly in \(\theta\). The nonrandom terms in the last two summands reduce to zero upon integrating by parts. The uniform convergence of the remaining terms is apparent.

Lemma 4.2

The formula on the next page gives \(\sigma^2\) for the classes S1 and S2, where \(\pi = 1 - \Phi\) for S1 tests and \(\pi = \alpha\) for S2 tests.
\[ \sigma^2 = \frac{1}{pq} \left\{ 2 \int_0^T \beta(s) \lambda(s) d\pi(s) \right\}^2 - \int_0^T \beta(s) \lambda(s) d\pi(s) \\
+ \int_0^T \beta(s) \lambda(s) d\pi(s) \left[ \int_0^T \beta(s) \lambda(s) d\pi(s) \right] \\
+ \int_0^T \int_0^T \lambda(s) d\pi(s) \beta(s) d\pi(s) \right\} \\
+ 2 \int_0^T \int_0^T \beta(T) \beta(s) d\pi(s) - \int_0^T \beta(T) \lambda(s) d\pi(s) \right\} \\
+ \int_0^T \beta(T) (1 - \beta(T)) - \left[ \int_0^T \beta(T) \right]^2 \right\}. \]

The variance \( \sigma^2 \) is obtained by substitution of

\[ q(t) = \frac{d}{dt} \int_0^T (\pi(t)) I\{0 \leq t \leq T\} + \int_0^T (\pi(t - 2T)) \lambda(2T) I\{2T \leq t \leq 3T\} \]

and \( q^* = \int_0^T (\pi(t)) \) in (3.8). The appendix contains the algebraic details.

**Example 4.1: Generalized Savage Test or Logrank Test**

This test is a member of both S1 and S2, because \( j(u) = 1 \).

This test is closely related to tests proposed by Cochran (1954), Mantel and Haenszel (1959), Peto and Peto (1972), and Cox (1972). Mantel and Haenszel propose a hypergeometric variance estimate, Cochran and Cox suggest a slightly different variance, and Peto and Peto refer to a permutation distribution. Using either S1 or S2,
\[ \mu_L(\theta) = \int_0^T \frac{\alpha(t) \alpha_0(t)}{\gamma(t)} (\lambda(t) - \lambda_0(t)) dt \]

and

\[ pq \sigma_L^2 = 2 \left[ \int_0^T \beta(t) \lambda(t) dt + \int_0^T \int_0^T \lambda(t) dt d\beta(s) \right] \]
\[ + 2\beta(T) \left[ \int_0^T d\beta(s) - \int_0^T \lambda(s) ds \right] \]
\[ + \beta(T)(1 - \beta(T)) - \beta^2(T) \]
\[ = 2\beta(T)\Lambda(T) - 2\beta(T)[\beta(T) - \Lambda(T)] + \beta(T) - 2\beta^2(T) \]
\[ = \beta(T) = p \text{ (Observe a death less than } T) \]

from Lemma 4.2. These formulae are equivalent to those deduced from Crowley (1974) (who discusses the case in which group membership may change once in time).

**Example 4.2: Generalized Wilcoxon Tests**

The first extension of the Wilcoxon test to allow censored observations was that proposed by Gehan (1965) and Gilbert (1962).

Tarone and Ware (1975) show that Gehan's test is an S2 test statistic with \( J(u) = 1 \), or \( j(u) = u \). Substitution of this identification into Table 4.1 and Lemma 4.2 show that

\[ \mu_G(\theta) = \int_0^T \alpha(t) \alpha_0(t)(\lambda(t) - \lambda_0(t)) dt = \mu_n(\theta) \]

and
\[ \sigma_G^2 = 2 \left[ \int_0^T \alpha^2(t) \beta(t) \lambda(t) dt - \int_0^T \beta(s) \alpha(s) \, ds \right] \\
+ \int_0^T \int_0^S \alpha(t) \lambda(t) dt \beta(s) ds \alpha(s) + \int_0^T \int_0^S \alpha(t) \lambda(t) dt \alpha(s) d\beta(s) \]

\[ + 2 \alpha(T) \beta(T) \left[ \int_0^T \beta(s) ds \alpha(s) + \int_0^T \alpha(s) d\beta(s) - \int_0^T \alpha(s) \lambda(s) ds \right] \\
+ \alpha^2(T) \beta(T) (1 - \beta(T)) - \alpha^2(T) \beta^2(T) = \int_0^T \alpha^2(s) d\beta(s). \]

The formulae are equivalent to those of Gehan (1965) for the exponential scale family and those of Breslow (1970) reduced from \( k \) samples to 2 samples.

Peto and Peto (1972) derived another extension of the Wilcoxon test. This latter extension is also obtained by Prentice (1978). (As with generalized Savage tests, there are several variants, depending on how the variability of the test is assessed.) Prentice and Marek (1979) show that this statistic is the SI test with \( j(u) = u \). The appropriate quantities for this statistic are

\[ \mu_P(\theta) = \int_0^T \frac{\alpha(t) \alpha_\theta(t)}{1 - H(t)} \left[ \lambda(t) - \lambda_\theta(t) \right] dt \]

and

\[ \sigma_P^2 = \int_0^T (1 - F(s))^2 \, d\beta(s). \]

The variance is computing by substituting \( \pi = 1 - F \) for \( \alpha \) in the computation of \( \sigma_G^2 \).
Example 4.3: Exponential Location Family

The family of translated exponential distributions
\[ F_\theta(t) = 1 - e^{-(t - \theta)_+} \]
is a non-contiguous family of alternatives.

All of the tests discussed in this paper have asymptotic efficiency zero relative to the tests designed for this specific family. It is nonetheless possible to compare the fairly standard tests described here.

Since the hazard rate of \( F_\theta \) is either 0 or 1, B1 is satisfied.

The condition B2 is implied by \( |F_\theta(t) - F(t)| \leq 1 - e^{-\theta} \). Since the family is a location family, (a) of Corollary 2.1 holds, and B3 is met. (Since \( \omega_n,\theta(t)\alpha_\theta(t) = \alpha_\theta(t)/\gamma_n,\theta(t) \) for the logrank test and \( \omega_n,\theta(t)\alpha_\theta(t) = \alpha_\theta(t) \) for the S2 Wilcoxon and

\[ 1 - [pF(t) + qF_\theta(t)] \]

for the S1 Wilcoxon, it is not difficult to check A5 and A6 directly.) Therefore the formulae of the preceding examples can be applied to obtain efficiencies.

Since \( \alpha_\theta(t)(\lambda_\theta(t) - \lambda(t)) = -I[0 < t < \theta] \) and

\[ \alpha_\theta(t)/\gamma_\theta(t) = 1 - H(t), \mu_L(t) = \mu_G(t) = \mu_P(t) = 1. \]

Therefore the efficacies of the tests are the reciprocals of the corresponding variances:

\[ pq\sigma_L^2 = \beta(T) = \int_0^T \alpha(t)\lambda(t)dt = \int_0^T (1 - H(t))e^{-t}dt \]

\[ pq\sigma_G^2 = \int_0^T \alpha^2(t)\lambda(t)dt = \int_0^T (1 - H(t))^3e^{-3t}dt \]

\[ pq\sigma_P^2 = \int_0^T (1 - F(t))^2\alpha(t)\lambda(t)dt = \int_0^T (1 - H(t))e^{-3t}dt \]

Since \( \sigma_G^2 \leq \sigma_P^2 \leq \sigma_L^2 \), when testing against this family of alternatives, the S2 Wilcoxon test is more efficacious than the S1
Wilcoxon test, which is in turn more efficacious than the logrank test. (This ordering reverses that obtained for the exponential scale family.) If \( H(t) \) is set equal to \( 1 - e^{-\nu t} \) (for convenient calculations), then

\[
\sigma_L^2(\nu) = \frac{1 - e^{-(1 + \nu)T}}{pq(1 + \nu)}
\]

\[
\sigma_G^2(\nu) = \frac{1 - e^{-3(\nu + 1)T}}{pq3(\nu + 1)}
\]

\[
\sigma_P^2(\nu) = \frac{1 - e^{-(3 + \nu)T}}{pq(\nu + 3)} = \sigma_G^2(\nu/3)
\]

Thus, for the translated exponential family, in the presence of exponential censoring, the S2 Wilcoxon test has the same efficacy as the S1 Wilcoxon test in the presence of censoring three times as intense.
V. DISCUSSION

This paper approached the problem of describing the behavior of test statistics for fixed alternatives by representing $T_n$ in terms of random processes $N_n^{(1)}$ and $N_n^{(2)}$, which are independent of the alternative, and transformations $T_0$, which are smoothly related to $\theta$. The technical tools which aided this endeavor are the properties of empirical measures as described by Dudley (1978) and the martingale methods, first applied in this area by Aalen. The advantages of this approach were illustrated by computing the Pitman efficiencies of two classes of linear rank tests for censored samples.

Another advantage of this approach is that it can be extended to other situations and other kinds of tests. The two samples were assumed to be exposed to equal censoring; a modification of the transformation $T_0$ would permit extensions of this case. This approach should permit study of a time-varying defined covariate in one group. In preliminary notes, the extension of this approach has been used to show that the normalized Kaplan-Meier estimator of $F_0$ converges uniformly in $\theta$ to its weak limit. This implies that it should be possible to extend the efficiency calculation to statistics based on the Kaplan-Meier estimator, such as the median test of Breslow and Crowley (1980). It may also be possible to allow random left censoring in addition to random right censoring.
It appears that k-sample tests can also be studied. Working with large k and families parametrized by a small number of parameters may lead to insight into the properties of various methods of adjusting for covariates.

A last advantage of this approach is that it gives an implicit formula for the variance of the test under a fixed alternative. It may therefore be possible to obtain approximate confidence intervals for parameters of interest.
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REFERENCES


The expansion of $n^{1/2}(T_n^2 - \mu_2(\theta))$ referred to on page 15 is

$$n^{1/2}(T_n^2 - \mu_2(\theta)) = \int_0^T [n^{1/2}(\mathbb{W}_{n,t}\theta(t) - \omega_{n,t}\theta(t)) - Z_{n,t}\theta(t)]$$

$$+ \int_0^T Z_{n,t}\theta(t)\left[\frac{N_1(\mathbb{W}_t)}{n_1} - \eta(\mathbb{W}_t)\right] \left[-\frac{dN_2(\tau_{B_t})}{n_2} - a_{n,t}\theta(t)dt\right]$$

$$+ \int_0^T Z_{n,t}\theta(t)\left[\frac{N_1(\mathbb{W}_t)}{n_1} - \eta(\mathbb{W}_t)\right] \frac{a_{n,t}\theta(t)dt}{n_1}$$

$$+ \int_0^T Z_{n,t}\theta(t)\eta(\mathbb{W}_t) \left[-\frac{dN_2(\tau_{B_t})}{n_1} - a_{n,t}\theta(t)dt\right]$$

$$+ \int_0^T Z_{n,t}\theta(t)\eta(\mathbb{W}_t) \frac{a_{n,t}\theta(t)dt}{n_2}$$

$$+ \int_0^T \omega_{n,t}\theta(t)\left[n^{1/2}\left(-\frac{N_1(\mathbb{W}_t)}{n_1} - \eta(\mathbb{W}_t)\right)\right] \frac{a_{n,t}\theta(t)dt}{n_2}$$

$$+ \int_0^T \omega_{n,t}\theta(t)\left[n^{1/2}\left(-\frac{N_1(\mathbb{W}_t)}{n_1} - \eta(\mathbb{W}_t)\right)\right] \frac{a_{n,t}\theta(t)dt}{n_2}$$

$$+ n^{1/2}\eta(\mathbb{W}_t)\omega_{n,t}\theta(T)\left[-\frac{N_2(\tau_{B_t})}{n_2} - \int_0^T a_{n,t}\theta(t)dt\right]$$
\[
- \int_0^T n^{1/2} \left( \frac{n^{(2)}_n(t^* R_T)}{n_2} \right) \int_0^T \frac{a_n \theta(s) ds}{n_2} d(\omega_n, \theta(t) \eta(V_t))
+ \int_0^T \omega_n \theta(t) \eta(V_t) n^{1/2} \left( \frac{a_n \theta(t)}{n_2} - a_\theta(t) \right) dt.
\]

The algebraic details for Lemma 4.2 are given below.

(A) \( \sigma^2 = \int_0^T \int_0^T q(s)K(s,t)q(t)dt ds - 2q^* \int_0^T K(s,T)q(s)ds + (q^*)^2K(T,T). \)

The first integral is expanded:

\[
\int_0^T \int_0^T q(s)K(s,t)q(t)dt ds = \int_0^T \int_0^T j'(\pi(s))[\beta(s\lambda t) - \beta(s)\beta(t)]
\]

\[
+ j'(\pi(s))d\pi(s)d\pi(t)
+ 2 \int_0^T \int_0^T j'(\pi(s))[(\beta(s) - \beta(t))^2 - \alpha(t)\beta(s)]j(\pi(t))\lambda(t)dt d\pi(s)
+ \int_0^T \int_0^T j(\pi(s))\lambda(s)[\alpha(s\lambda t) - \alpha(s)\alpha(t)]j(\pi(t))\lambda(t)dt ds
\]
Each of the terms can be reworked:

\[
\int_0^T \int_0^T j'(\pi(s)) [\beta(s\Delta t) - \beta(s)\beta(t)] j'(\pi(t)) \, d\pi(s) \, d\pi(t)
\]

\[
= 2 \int_0^T j'(\pi(s)) \beta(s) [\int_0^T j'(\pi(t)) \, d\pi(t)] \, d\pi(s) - [\int_0^T j'(\pi(s)) \beta(s) \, d\pi(s)]^2
\]

\[
= 2 \int_0^T j'(\pi(s)) \beta(s) [j(\pi(T)) - j(\pi(s))] \, d\pi(s) - [\int_0^T j'(\pi(s)) \beta(s) \, d\pi(s)]^2
\]

\[
= 2j(\pi(T)) \int_0^T j'(\pi(s)) \beta(s) \, d\pi(s) - 2 \int_0^T j'(\pi(s)) \beta(s) j(\pi(s)) \, d\pi(s)
\]

\[- [\int_0^T j'(\pi(s)) \beta(s) \, d\pi(s)]^2.\]

\[
2 \int_0^T \int_0^T j'(\pi(s)) [(\beta(s) - \beta(t))_+ - \alpha(t)\beta(s)] j(\pi(t)) \lambda(t) \, dt \, d\pi(s)
\]

\[
= 2 \int_0^T \int_0^S j'(\pi(s)) (\beta(s) - \beta(t)) j(\pi(t)) \lambda(t) \, dt \, d\pi(s)
\]

\[- 2 \int_0^T j'(\pi(s)) \beta(s) \, d\pi(s) \int_0^T \alpha(t) j(\pi(t)) \lambda(t) \, dt
\]

\[
= 2 \int_0^T j'(\pi(s)) \beta(s) (\int_0^S j(\pi(t)) \lambda(t) \, dt) \, d\pi(s)
\]

\[- 2 \int_0^T j'(\pi(s)) \int_0^S j(\pi(t)) \beta(t) \lambda(t) \, dt \, d\pi(s)
\]

\[- 2 \int_0^T j'(\pi(s)) \beta(s) \, d\pi(s) \int_0^T j(\pi(t)) \, d\beta(t).\]
\[ \int_0^T \int_0^T j(\pi(s))\lambda(s)[\alpha(s)\nu(t) - \alpha(s)\alpha(t)]d\eta(t)\lambda(t)dt\, ds \]

\[ = 2 \int_0^T j(\pi(s))\lambda(s)\alpha(s) \int_0^s j(\pi(t))\lambda(t)dt\, ds - (\int_0^T j(\pi(s))\alpha(s)\lambda(s)ds)^2 \]

\[ = 2 \int_0^T j(\pi(s)) \int_0^s j(\pi(t))\lambda(t)dt\, d\beta(s) - (\int_0^T j(\pi(s))d\beta(s))^2. \]

Similarly, the second integral in (A) is

\[ \int_0^T k(s,T)q(s)ds = \int_0^T j'(\pi(s))\beta(s)(1 - \beta(T))d\pi(s) \]

\[ + \int_0^T j(\pi(s))\lambda(s)[\beta(T) - \beta(s) - \alpha(s)\beta(T)]ds \]

\[ = (1 - \beta(T)) \int_0^T j'(\pi(s))\beta(s)d\pi(s) + \beta(T) \int_0^T j(\pi(s))\lambda(s)ds \]

\[ + \int_0^T j(\pi(s))\beta(s)\lambda(s)ds - \beta(T) \int_0^T j(\pi(s))d\beta(s). \]

The third term of (A) is \( (q^*(T))^2 k(T,T) = j^2(\pi(T))\beta(T)(1 - \beta(T)). \)
Substituting,

\[ \sigma^2 = 2 \left \{ j(\pi(T)) \int_0^T j'(\pi(s)) \beta(s) \, d\pi(s) - \int_0^T j'(\pi(s)) \beta(s) j(\pi(s)) \, d\pi(s) \right \}

+ \int_0^T \beta(s) j'(\pi(s)) \left \{ \int_0^T j(\pi(t)) \lambda(t) \, dt \right \} \, d\pi(s)

- j(\pi(T)) \int_0^T \beta(t) j(\pi(t)) \lambda(t) \, dt + \int_0^T j^2(\pi(t)) \beta(t) \lambda(t) \, dt

+ \int_0^T j(\pi(s)) \int_0^T j(\pi(t)) \lambda(t) \, dt \, d\beta(s)

- j(\pi(T)) (1 - \beta(T)) \int_0^T j'(\pi(s)) \beta(s) \, d\pi(s)

- j(\pi(T)) \beta(T) \int_0^T j(\pi(s)) \lambda(s) \, ds + j(\pi(T)) \int_0^T j(\pi(s)) \beta(s) \lambda(s) \, ds

+ j(\pi(T)) \beta(T) \int_0^T j(\pi(s)) \, d\beta(s) \}

+ j^2(\pi(T)) \beta(T) (1 - \beta(T)) - \left \{ \int_0^T j(\pi(s)) \beta(\pi(s)) \, ds + \int_0^T \beta(s) j'(\pi(s)) \, d\pi(s) \right \}^2

= 2 \left \{ \int_0^T j^2(\pi(s)) \beta(s) \lambda(s) \, ds - \int_0^T j'(\pi(s)) \beta(s) j(\pi(s)) \, d\pi(s) \right \}

+ \int_0^T \beta(s) j'(\pi(s)) \left \{ \int_0^T j(\pi(t)) \lambda(t) \, dt \right \} \, d\pi(s)

+ \int_0^T \int_0^T j(\pi(t)) \lambda(t) \, dt \, j(\pi(t)) \, d\beta(s) \}

+ 2 j(\pi(T)) \beta(T) \left \{ \int_0^T j'(\pi(s)) \beta(s) \, d\pi(s) - \int_0^T j(\pi(s)) \lambda(s) \, ds

+ \int_0^T j(\pi(s)) \, d\beta(s) \}

+ j^2(\pi(T)) \beta(T) (1 - \beta(T)) - \left \{ \int_0^T \lambda(\pi(s)) \beta(s) \right \}^2. \quad \text{QED.}