Bias point of estimation following a group sequential test

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Abstract

Repeated significance testing in a sequential experiment not only increases the overall type I error rate of the false positive conclusion, but also causes the bias in estimating the unknown parameter. In this paper we investigate the biases, both conditioning upon the stopping time and in overall, in estimating the drift parameter of a Brownian motion process with interim monitoring rule. It is found out that the conditional bias may be very serious for existing point estimation methods, even though their unconditional bias can still be satisfactory. Methodology of several new conditional estimators which can significantly reduce the conditional bias from unconditional estimators is proposed. The results of Monte-Carlo studies comparing the proposed conditional estimation methods with some existing methods are also presented in this paper.

1 Introduction

Group sequential designs have been often adopted in hypothesis testing experiments such as clinical trials in order to reduce the expected sample size and obtain a significant result before the scheduled end of the studies. In many cases the observed statistics have (asymptotic) Gaussian independent increment structure under the null hypothesis and furthermore they often behave like a Brownian motion with a linear drift under alternatives in an information time scale (Tsiatis (1982), Sellke &
Siegmund (1983), Slud (1984) Lan & Zucker (1993)). So it’s more general and convenient to use the continuous Brownian motion process with interim stopping rule to represent an actual sequential analysis. In this paper we use $B(t)$ to denote a standard Brownian motion process over $t \in [0, 1]$ and $S_\theta(t) = B(t) + \theta t$ is a Brownian motion with a linear drift where the drift parameter $\theta$ is usually directly associated with the original hypothesis parameter. In trials to establish a treatment effect, the null hypothesis generally is $H_0: \theta = 0$ and the alternative $H_A$ is $\theta \neq 0$ or $\theta > (<)0$ in a one-sided case.

Group sequential designs were first introduced to cope with the inflation of the overall type I error due to the repeated testing (Pocock (1977), O’Brien & Fleming (1979)). Different variations of group sequential boundaries and $\alpha$-spending functions (Lan & DeMets (1983)) which is a more flexible and unifying method are later developed. However, despite the fact that the group sequential design has been well adapted to hypothesis testing, there is another concern which is the potential bias of the maximum likelihood estimator (MLE) of the parameter $\theta$, denoted by $\hat{\theta}$. Because of the dependence between the true parameter and stopping time, it is known that the MLE $\hat{\theta} = S(t)/t$ will be biased toward extreme on average (e.g. Kim (1995), Pinheiro & DeMets (1995)). So to find an appropriate estimator in a sequential experiment becomes more complicated. Many researchers have studied such bias problem and explored different kinds of estimators. Those include Whitehead’s bias reduced method (1986), median unbiased estimator by Kim (1989), Emerson and Fleming (1990) and an unbiased estimator by Emerson (1993). Those estimators have been shown to have smaller bias than the MLE. Interestingly, the bias refered to is only the marginal bias of an estimator. On the other hand, since the stopping time is observed when a group sequential study is finished it might be more relavent and make more sense to look at the bias conditioning on stopping time as well.

In Section 2 we study the conditional bias of the MLE $\hat{\theta}$ in additional to its unconditional
bias in a hypothetical example and find the conditional bias to be more problematic. Depending on the stopping time, the \( \hat{\theta} \) can be biased toward either the null hypothesis or the extreme. In Section 3 we propose a maximum conditional likelihood estimator (MCLE) aiming to reduce the conditional bias from \( \hat{\theta} \). It is also demonstrated that the MCLE is equivalent to a conditional moment estimator and a conditional biased reduced estimator. In addition, two modified hybrid estimators are proposed. A “future independence” concept is also introduced which is desirable for sequential estimation. Afterwards, in Section 4 we compare the MLE, the Whitehead’s bias adjusted estimator and conditional estimators under several different group sequential settings using the Monte-Carlo simulation method. At the end some further discussions in Section 5.

2 Conditional and Unconditional Bias of the MLE

Suppose there are \( M \) interim looks scheduled at \( 0 < t_1 < \cdots < t_M = 1 \). For the ease of notation, the observed \( S_{\theta}(t_k) \) is abbreviated as \( S_k, k = 0, 1, \ldots, M \). Define \( X_k := S_k - S_{k-1} \), is the increment of the \( S_{\theta}(t) \) from time \( t_{k-1} \) to \( t_k \). Denote \( (b^*_k, c^*_k) \) the acceptance region for \( S_k, \ k = 1, 2, \cdots, M \), where the endpoints can be \( \pm \infty \). \( A_k = (b^*_k, c^*_k) \) if \( k < M \) and the empty set if \( k = M \) is the continuation region for \( S_k \). The stopping stage \( \eta := \min\{k \leq M : S_k \notin A_k\} \) where \( t_\eta \) is the time to stop the process \( S_\theta(t) \). As a reminder, \( \hat{\theta} = S_\eta/t_\eta \) is the usual maximum likelihood estimator (MLE) of the parameter \( \theta \) which takes the same form as in a non-sequential test. We use \( b(\hat{\theta}) := E_\theta(\hat{\theta}) - \theta \) to denote the unconditional bias of \( \hat{\theta} \) and similarly \( b(\hat{\theta}|\eta) := E_\theta(\hat{\theta}|\eta) - \theta \) for the bias conditioning on the stopping stage \( \eta \), or more precisely conditioning on the event \( \{X_\eta \notin A_\eta, \ X_k \in A_k, \ k < \eta\} \).

To demonstrate the difference between the conditional and unconditional biases of the MLE, we will first use the following hypothetical example of a group sequential design. Let’s consider a symmetric two-sided O’Brien-Fleming (OBF) type group sequential boundary at the overall level \( \alpha = 0.05 \), which is calculated from an alpha spending function \( \alpha_1(t) = 2 - 2\Phi(Z_{\alpha/2}/\sqrt{t}) \), with
five interim looks to monitor the Brownian motion process \( S_\theta(t) \) at time 0.2, 0.4, 0.6, 0.8 and 1.0.

**Figure 2.1** displays the conditional biases \( b(\hat{\theta}|\eta = k) \), \( k=1,2,3,4,5 \) and the marginal bias of \( \hat{\theta} \). The plots are generated by using numerical calculation of the conditional densities \( S_\theta(t_k|\eta = k) \), \( k=1,2,3,4,5 \).

![Graphs showing conditional biases](image)

*Figure 2.1: Plots of the conditional bias functions \( b(\hat{\theta}|\eta) \) and the marginal bias function \( b(\hat{\theta}) \) for a 2-sided OBF type boundary at \( T=(.2,.4,.6,.8,1) \). The X axis is the true drift parameter \( \theta \) and the Y axis shows bias.*

In these plots only the areas where \( \theta \geq 0 \) are shown. The other halves of the bias graphs for \( \theta < 0 \) are omitted because the bias is symmetric and centered at \( \theta = 0 \). We are most interested in the areas where the value of \( \theta \) smaller than 5 or 6 so that the power is below 99.9%, which
is usually the case in practice. The larger values of \( \theta \) are presented mostly for the purpose of theoretical interest. From these graphs, we can see that although the overall bias only ranges from 0 to 0.45, depending on the true value of the \( \theta \), the conditional bias can be as high as 9 or as low as -5. As the trial is stopped at different interim stages the biases \( b(\hat{\theta} | \eta) \)'s also differ dramatically. Similar phenomenon of different bias patterns in term of risk ratio was also observed by Hughes and Pocock (1988) but they did not investigate the conditional bias in further depth. Compared to the conditional biases, the overall bias \( b(\hat{\theta}) \) at the lower right corner does not look impressive at all. In fact, from the property of conditional expectation,

\[
\begin{align*}
b(\hat{\theta}) &= E_{\theta}(\hat{\theta}) - \theta \\
&= E[E_{\theta}(\hat{\theta} | \eta) - \theta] \\
&= E[b(\theta | \eta)] \\
&= \sum_k b(\theta | \eta = k) P(\eta = k)
\end{align*}
\]

which shows that the overall bias is a weighted average of all the conditional biases with the weight function being the stopping probability at each interim stage.

From the plots in Figure 2.1, the overall bias is always positive (except for \( \theta = 0 \)) and converges to zero as the \( \theta \) goes to infinity. That doesn't apply to the conditional biases except for the first interim (\( \eta = 1 \)). The asymptotic values of \( b(\hat{\theta}) \) and \( b(\hat{\theta} | \eta = 1) \) are same because \( \lim_{\theta \to \infty} P(\eta = 1) = 1 \); that is, for sufficiently large \( \theta \) the trial will almost for sure be stopped at the first stage. On the other hand, the bias of the MLE is always negative when stopped at the last stage (\( \eta = M \)). Those are general results under the symmetric group sequential boundary and usually the case even under an asymmetric monitoring boundary.

The previous observations for \( \eta = 1 \) are stated as the Theorem 1 below. The conclusion for
\( \eta = M \) is proved as corollary 2.2 of Theorem 2. In addition, except for the case \( \eta = M \), the conditional bias functions have a right-skewed uni-modal shape which grows from zero and after the peak drops toward zero when \( \eta = 1 \) or toward negative when \( \eta \) is between 1 and \( M \).

**Theorem 1** Using the notation as defined before, i.e. let \( S_\eta(t) = B(t) + \theta t \), \( t \in [0,1] \) denote a standard Brownian motion with drift parameter \( \theta \) and \( M(\geq 2) \) interim looks at \( 0 < t_1 < t_2 < \cdots < t_M = 1 \). Suppose the first continuation region \( A_1 \) for \( S_1 \) is an finite interval \( (-c_1^*, c_1^*) \) centering at zero and \( S_\eta(t) \) is stopped at the first stage, i.e. \( \eta = 1 \), then we have

\[
|E_\theta(\hat{\theta}|\eta = 1)| > |\theta|
\]

for \( \theta \neq 0 \) and

\[
\lim_{\theta \to \infty} b(\hat{\theta}|\eta = 1) - \theta = 0
\]

See proof of the theorem in Appendix A.1.

**Theorem 2** Notation as defined before and assuming the parameter \( \theta > 0 \). Suppose the continuation regions \( A_k \)'s are finite intervals \( (b_k^*, c_k^*) \) for \( k = 1, 2, \ldots, M - 1 \) and \( S_\eta(t) \) reaches the final interim, i.e. the stopping stage \( \eta = M \). If the centers of \( A_k \)'s, \( a_k = (b_k^* + c_k^*)/2 \) satisfy that \( a_k < \theta t_k \) for \( k = 1, \ldots, M - 1 \), then the usual MLE \( \hat{\theta} \) at the last interim stage underestimates the true \( \theta \) on average, i.e.

\[
E_\theta(\hat{\theta}|\eta = M) < \theta \tag{2.1}
\]
Since $\hat{\theta} = S_\eta/t_\eta$, it’s obvious that at the last interim stage $\hat{\theta} = S_M$. Therefore

$$E_\theta(\hat{\theta}|\eta = M) = E_\theta(S_M|\eta = M) = E_\theta(S_{M-1} + X_M | \eta = M)$$

$$= E_\theta(S_{M-1} | \eta = M) + \theta(1 - t_{M-1}).$$

So it’s equivalent for the above theorem to show

$$E_\theta(S_{M-1} | \eta = M) < \theta t_{M-1} \quad (2.2)$$

and we will use the following lemma to prove it.

**Lemma 2.1** Under the assumptions in Theorem 2, suppose $\theta = 0$, that is, $S_\theta(t) = B(t)$ which is a standard Brownian motion and $a_k = (b_k^* + c_k^*)/2 < 0$, $k = 1, \ldots, M - 1$. Let function $f_k$ denote the density of $S_k$, $k = 1, \ldots, M - 1$ and $\text{I}$ denote an indicator function. Then

$$f_{M-1}(s) \cdot I_{A_{M-1} \cap (-\infty, 0)}(s) \geq f_{M-1}(-s) \cdot I_{A_{M-1} \cap (0, \infty)}(-s) \quad (2.3)$$

for $s < 0$ with the strict inequality holds in a non-degenerated interval.

See proof of the Lemma in Appendix A.2. \hfill \Box

**Proof of Theorem 2:**

Since $B(t) = S_\theta(t) - \theta t$, we can transform the random process $S_\theta(t)$ to $B(t)$ by shifting the drift by $-\theta$. Then the new equivalent continuation intervals for $B(t)$ are $A_k^* = A_k - \theta t_k = (b_k^* - \theta t_k, c_k^*- \theta t_k)$, $k = 1, \ldots, M - 1$. It’s easy to verify that the assumptions in Lemma 2.1 hold for
\( B(t) \). Therefore

\[
E_{\theta} |B(t_{M-1})| \eta = M \cdot B(t_{M-1}) = \int_{A_{M-1}} \cdot f_{M-1}(s) ds
\]

\[
= \int_{A_{M-1} \cap (-\infty,0)} + \int_{A_{M-1} \cap [0,\infty)} s \cdot f_{M-1}(s) ds
\]

\[
= \int_{-\infty}^{\infty} s[f_{M-1}(s) \cdot I_{A_{M-1} \cap (-\infty,0)}(s) + f_{M-1}(-s) \cdot I_{A_{M-1} \cap [0,\infty)}(-s)] ds
\]

\[
< 0
\]

The inequality (2.2) and conclusion of the theorem simply follow. \( \square \)

The result of Theorem 2 can be easily extended to:

**Corollary 2.1** In Theorem 2, the assumptions imposed on the continuation intervals \( A_k \)'s can be relaxed to

1. the lower bounds \( b_k^* \), \( k = 1, \ldots, M - 1 \) can be \(-\infty\);

2. \( a_k \leq \theta t_k \), \( k = 1, \ldots, M - 1 \) where at least one of the inequalities strictly holds.

**Corollary 2.2** (Notation as before) Suppose symmetric sequential boundaries are used to monitor \( S_\theta \) and it reaches the final interim, i.e. \( \eta = M \) or \( t_M = 1 \), the following inequalities hold for any \( \theta \neq 0 \):

\[
0 < \text{sign}(\theta) \cdot E_{\theta} (\hat{\theta} | \eta = M) < |\theta|
\]

**Proof:**
Without loss of generality, assume $\theta > 0$. Because symmetric boundaries are applied, i.e. $b_k^* = -c_k^*$, $A_k$'s are centered at zero for $k = 1, 2, \ldots, M$. From Theorem 2, $E_0(\hat{\theta}|\eta = M) < \theta$.

Also because of the strict monotonicity of $E_0(\hat{\theta}|\eta = M)$ in $\theta$ which will be proved later in equation (3.8), $E_0(\hat{\theta}|\eta = M) > E_0(\hat{\theta}|\eta = M) = 0$ and the conclusion follows.

Whitehead (1986) proposed a bias reduced estimator, denoted by $\tilde{\theta}$, which was further explored in the papers of Emerson & Fleming (1990), Pinheiro & DeMets (1997) and Li & DeMets (1996). The bias reduction is obtained by subtracting an estimate of the unconditional bias $b(\hat{\theta})$ from the $\hat{\theta}$ and hence the overall bias is reduced. However, this approach ignores the different patterns of the bias under different stopping stages and treats them equally as the marginal bias. As a result the $\tilde{\theta}$ is closer to zero than the MLE $\hat{\theta}$ but still has large bias when stopping early. And actually it may exaggerate the underestimation when the conditional bias of $\hat{\theta}$ is negative, mostly at the last stage. As being pointed out before, the overall bias of the $\tilde{\theta}$ is a weighted average of individual conditional biases. Also because the positive conditional biases at earlier stopping stage are nearly offset by the negative ones at the later stopping stages, the overall bias can turn out to be negligible. Therefore, it’s somewhat misleading to only look at the overall bias and reducing the overall unconditional bias alone may not be enough. Since the overall bias is an average of all conditional biases, the $\hat{\theta}$ then depends on every planned interim even those after the stopping time. This is unintuitive because when early stopping happens the future monitoring criterion hasn’t been realized and thus shouldn’t affect the estimation. In general, an ideal estimator should shrink from the $\hat{\theta}$ more in the trials stopped early and less in the trials stopped late. That desirable property has been pointed out by several statisticians (e.g. Pocock & Hughes (1989), Whitehead (1997)). After all, a good estimator should not only have small overall bias but also small conditional bias.

When estimating the treatment effect after a trial or experiment is stopped early, we already
know the interim analysis time, usually the last observed stage, at which the test statistic \( S_k \) first crossed the boundary. This piece of information should be taken into account and the \( \hat{\theta} \) be adjusted accordingly. In next section, we propose and explore a \textit{maximum conditional likelihood estimator} (MCLE), denoted by \( \check{\theta} \), which can reduce the bias conditionally and in overall as well. Results show that the MCLE has two equivalent forms of the conditional moment estimate and the conditional bias reduction estimate. Two modified versions of the MCLE to reduce its variance will also be presented and discussed.

3 Conditional Point Estimators

It has been shown that the doublet \((\eta, S_\eta)\) is the sufficient statistics of the drift parameter \( \theta \) (e.g. Chang (1989)). Since our objective is to minimize the conditional bias of an estimator given the stopping stage \( \eta \), we focus on drawing inference from the conditional distribution of \( S_\eta \) given \( \eta \). The following three conditional estimators turn out to be equivalent to each other.

3.1 Maximum conditional likelihood estimate (MCLE)

Denoted by \( \check{\theta}_1 \), the MCLE maximizes the likelihood of \((\eta, X_1, \ldots, X_\eta)\) conditioning on the stopping stage \( \eta = n \). Mathematically,

\[
\check{\theta}_1 = \arg\max_{\theta} f_\theta(\eta, X_1, \ldots, X_\eta | \eta = n) \\
= \arg\max_{\theta} \frac{f_\theta(X_1, \ldots, X_\eta, \eta = n)}{P_\theta(\eta = n)} \\
= \arg\max_{\theta} \frac{f_\theta(X_1, \ldots, X_\eta)P_\theta(\eta = n|X_1, \ldots, X_\eta)}{P_\theta(\eta = n)} \\
= \arg\max_{\theta} \frac{f_\theta(X_1, \ldots, X_\eta)}{P_\theta(\eta = n)} \\
= \arg\max_{\theta} L_{\text{cond}} \quad (3.1)
\]
where \( L_{\text{cond}} = \log (f_\theta (X_1, \cdots, X_n)) - \log (P_\theta (\eta = n)) \) and \( f_\theta \) is the notation representing density functions.

Equivalently, \( \hat{\theta}_1 \) is a root of the conditional likelihood equation:

\[
\frac{\partial}{\partial \theta} L_{\text{cond}} = \frac{\partial}{\partial \theta} \log (f_\theta (X_1, \cdots, X_n)) - \frac{\partial}{\partial \theta} \frac{P_\theta (\eta = n)}{P_\theta (\eta = n)} = 0
\]  

(3.2)

Note that the conditional log-likelihood \( L_{\text{cond}} \) takes the form of a penalized log-likelihood function with the penalty function \( \log P_\theta (\eta = n) \). If a symmetric sequential boundary is used and the observed \( \eta = 1 \), because the \( P_\theta (\eta = 1) \) is monotone increasing in the positive values of \( \theta \), the likelihood will be penalized more for larger \( \theta \) than for smaller \( \theta \) (in absolute value). Thus \( \hat{\theta}_1 \) will be closer to zero than \( \hat{\theta} \) which is what we desire. On the other hand, \( P_\theta (\eta = M) \) is monotone decreasing in the positive values of \( \theta \) so if \( \eta = M \), \( \hat{\theta}_1 \) will be farther from zero than \( \hat{\theta} \). That's also favorable to us because we know that the MLE \( \hat{\theta} \) will underestimate the true \( \theta \) when the process \( S_\theta (t) \) reaches the end at \( t = 1 \).

Since \( S_\theta (t) \) under the interim analyses is a restricted process from Brownian motion, the conditional likelihood \( L_{\text{cond}} \) doesn't have a closed form so it's not very simple to calculate the maximizer of \( L_{\text{cond}} \) or the solution to equation (3.2) even using numerical methods. However, the following equivalent estimator gives us an alternative approach.

### 3.2 Conditional moment estimate

Denoted by \( \hat{\theta}_2 \), the CME is the solution of \( \theta \) to the conditional moment estimate equation:

\[
E_{\theta} [(t_\eta, S_\eta) | \eta = n] = (t_n, s_n)
\]  

(3.3)
or equivalently,

\[ E_\theta[S_\eta|\eta = n] = s_n \quad (3.4) \]

From the above equation, we have another equivalent form

\[ E_\theta[\hat{\theta}|\eta = n] = E_\theta[S_\eta/t_n|\eta = n] = s_n/t_n = \hat{\theta}_{n=n} \]

The next general theorem will show that the MCLE and CME are identical.

**Theorem 3** Suppose \(X_1, X_2, \ldots\) are sequentially and independently observed and each \(X_k\) is from an exponential family distribution \(p_{\theta,k}(x) = h_k(x) \exp(\lambda_k(\theta) \cdot T_k(x) - B_k(\theta))\). \(S_k = \sum_{i=1}^{k} X_i\) is the partial sum. The sampling will stop at \(X_\eta\) if \(S_k \in A_k\) for \(k < \eta\) and \(S_\eta \notin A_\eta\), where the \(A_k\)'s are any continuation regions for \(S_k\)'s such that \(\eta\) is bounded, i.e. \(P_\theta(\eta \leq M) = 1\) for some \(M\). Suppose we observe \(x_1, x_2, \ldots, x_n\) and \(\eta = n \leq M\). Then the solution to the conditional likelihood equation (3.2) is equivalent to the solution to the following equation:

\[ \sum_{i=1}^{n} \lambda_i'(\theta) \cdot E_\theta[T_i(X_i)|\eta = n] = \sum_{i=1}^{n} \lambda_i'(\theta) \cdot T_i(x_i) \quad (3.5) \]

Applying the above theorem to the special case of a Brownian motion with drift parameter, we can easily see that by plugging in the theorem \(T_i(x) = x\) and \(\lambda_i(\theta) = \theta\), the equation (3.5) turns out to be

\[ \sum_{i=1}^{n} E_\theta[X_i|\eta = n] = \sum_{i=1}^{n} x_i \]

which is exactly the equation (3.4).
To prove Theorem 3, a lemma is needed first. Based on the paper of Whitehead (1986), Pinheiro and DeMets (1997) have provided a very useful formula showing that the process \( S(t) \) satisfies:

\[
\frac{\partial}{\partial \theta} E_\theta [g(S(t_\eta), t_\eta)] = E_\theta [(S(t_\eta) - \theta t_\eta) g(S(t_\eta), t_\eta)]
\]  

(3.6)

where \( \eta \) is stopping time as defined before and \( g \) is a real function such that \( E_\theta[|g(S(t_\eta), t_\eta)|] < \infty \) for any \( \theta \).

An alternative proof of the formula (3.6) can be shown from the following more general results.

**Lemma 3.1** Under the same assumptions as in Theorem 3, and if a real function \( g \) satisfies that \( E_\theta[|g(S(t_\eta), t_\eta)|] < \infty \) for any \( \theta \), then the following equation holds

\[
\frac{\partial}{\partial \theta} E_\theta[g(S_\eta, t_\eta)] = E_\theta \left[ \sum_{i=1}^{\eta} T_i(X_i) \cdot \lambda'_i(\theta) - \sum_{i=1}^{\eta} B'_i(\theta) g(S_\eta, t_\eta) \right]
\]  

(3.7)

See proof of the Lemma in Appendix A.3.

Choose \( g(s, t) = I_{\{t=t_\eta\}} \), (\( I \) is indicator function), then

\[
E_\theta[g(S_\eta, t_\eta)] = E_\theta[I_{\{\eta=n\}}] = P_\theta(\eta = n)
\]
Using Lemma 3.1, the conditional likelihood equation at \( \eta = n \) can be written as

\[
0 = L_{\text{cond}} = \frac{\partial}{\partial \theta} \left\{ \log \left( f_\theta (x_1, \cdots, x_n) \right) - \log E_\theta[I_{(\eta=n)}] \right\} = \frac{\partial}{\partial \theta} \sum_{i=1}^n [T_i(x_i) \cdot \lambda_i(\theta) - B'_i(\theta)] - \frac{\partial}{\partial \theta} \frac{E_\theta[I_{(\eta=n)}]}{P_\theta(\eta = n)} = \sum_{i=1}^n \left[ T_i(x_i) \cdot \lambda'_i(\theta) - B'_i(\theta) \right] - \frac{E_\theta[(\sum_{i=1}^n T_i(X_i) \cdot \lambda'_i(\theta) - \sum_{i=1}^n B'_i(\theta))I_{(\eta=n)}]}{P_\theta(\eta = n)} = \sum_{i=1}^n \lambda'_i(\theta) \cdot T_i(x_i) - \sum_{i=1}^n \lambda'_i(\theta) \cdot E_\theta[T_i(X_i)|\eta = n]
\]

which is the conclusion of Theorem 3.

Before solving the estimation equation (3.4), two concerns naturally rise:

1. Does the equation (3.4) have a solution?

2. If it does, is the solution unique?

To answer the above two questions, we first notice that

\[
E_\theta[|S_\eta|] \leq E_\theta[\sum_{i=1}^M |X_i|] < \infty
\]

Therefore, by applying formula (3.6) again:

\[
\frac{\partial}{\partial \theta} E_\theta[S_\eta|\eta = n] = \frac{\partial}{\partial \theta} \left\{ \frac{E_\theta[S_nI_{(\eta=n)}]}{E_\theta[I_{(\eta=n)}]} \right\} = \frac{E_\theta[(S_n - \theta t_n)S_nI_{(\eta=n)}]E_\theta[I_{(\eta=n)}] - E_\theta[S_nI_{(\eta=n)}]E_\theta[(S_n - \theta t_n)I_{(\eta=n)}]}{\left\{E_\theta[I_{(\eta=n)}]\right\}^2} = \frac{E_\theta[S_n^2I_{(\eta=n)}]E_\theta[I_{(\eta=n)}] - (E_\theta[S_nI_{(\eta=n)}])^2}{\left\{E_\theta[I_{(\eta=n)}]\right\}^2} = E_\theta[S_n^2|\eta = n] - E_\theta[S_n|\eta = n]^2 = Var_\theta[S_n|\eta = n] > 0
\]

(3.8)
That is, \( E_\theta[S_n|\eta = n] \) is strictly increasing in \( \theta \) so there’ll be at most one root to equation (3.4). Next, to prove the existence of such root, it is not difficult to see that for any given \( S_{n-1} = s_{n-1} \in A_{n-1} \),

\[
\lim_{\theta \to \infty} E_{\theta}[X_n|\eta = n, S_{n-1} = s_{n-1}] = \infty
\]

and thus

\[
\lim_{\theta \to \infty} E_{\theta}[S_n|\eta = n, S_{n-1} = s_{n-1}] = s_{n-1} + \lim_{\theta \to \infty} E_{\theta}[X_n|\eta = n, S_{n-1} = s_{n-1}] = \infty
\]

where the above convergence is uniformly for \( s_{n-1} \in A_{n-1} = (b^*, c^*_{n-1}) \). So

\[
\lim_{\theta \to \infty} E_\theta[S_n|\eta = n] > s_n \quad (3.9)
\]

Likewise, in the other direction

\[
\lim_{\theta \to -\infty} E_\theta[S_n|\eta = n] < s_n \quad (3.10)
\]

Because \( E_\theta[S_n|\eta = n] \) is continuous in \( \theta \), the equation (3.4) has a root for any \( s_n \). That is, we’ve proved that the conditional estimator, \( CME \) or \( MCLE \), uniquely exists. Also because of the monotonicity of \( E_\theta[S_n|\eta = n] \) in \( \theta \), it might be easier to solve equation (3.4) using numerical methods than to solve the likelihood equation (3.2) once we can calculate the conditional expectation \( E_\theta[S_n|\eta = n] \) by either simulation or a numerical method. To confirm this, the conditional expectation \( E_\theta[\hat{\theta}|n] \) together with the marginal expectation \( E_\theta[\hat{\theta}] \) for different parameter values of \( \theta \) are plotted in Figure 3.1. All the bias curves are monotone which ensures a unique root of equation (3.4). Again, the overall expectation curve is the weighted average of the conditional expectation.
curves with the stopping probabilities as the weight functions. Although the overall expectation vs. the true $\theta$ is little distinguished from the 45° line, the conditional expectation curves are quite different.

![Graphs showing conditional expectations](image)

**Figure 3.1:** Plots of the conditional expectations $E_\theta(\hat{\theta}|\eta)$ and the marginal expectation $E_\theta(\hat{\theta})$ (the solid lines) for a 2-sided OBF type boundary at $T=(2,4,6,8,1)$. The X axis is the true drift parameter $\theta$ and the dots are 45 degree lines.

Following is an interesting result:

**Corollary 3.1** Let $\hat{\theta}_n$ denote the MLE of $\theta$ based only on the stopping stage $\eta = n$, i.e. $\hat{\theta}_n = \arg \max_{\theta} P_\theta(\eta = n)$. Then $\hat{\theta}_n$ is a root of the following equation (in $\theta$):
\[ E_\hat{\theta}[\hat{\theta}|\eta = n] = \theta \text{ or } b(\hat{\theta}|\eta) = 0 \]  

(3.11)

Proof:

Since \( \hat{\theta}_n = \arg \max_\theta P_\theta(\eta = n) \), \( \hat{\theta}_n \) satisfies

\[ \frac{\partial}{\partial \theta} P_\theta(\eta = n) = 0 \]

Plug in formula (3.6) with \( g = I_{(t_n = t_n)} = I_{(\eta = n)} \), we have

\[ 0 = \frac{\partial}{\partial \theta} E_\theta[I_{(\eta = n)}] = E_\theta[(S_n - \theta t_n)I_{(\eta = n)}] \]

\[ \iff E_\theta[S_nI_{(\eta = n)}] = \theta \cdot E_\theta[t_nI_{(\eta = n)}] \]

\[ \iff \theta = \frac{E_\theta[S_nI_{(\eta = n)}]}{t_nE_\theta[I_{(\eta = n)}]} = E_\theta[S_n|\eta = n]/t_n = E_\theta[\hat{\theta}|\eta = n] \]

which is exactly equation (3.11). \( \square \)

From Corollary 3.1, we know that if the true drift parameter \( \theta \) happens to be the same as the MLE of \( \theta \) based solely on the stopping stage \( \eta = n \), then the conditional bias of the usual MLE \( \hat{\theta} \) is exactly zero. Another \( \theta \) value with zero conditional bias is zero in a symmetric boundary case. For other values of \( \theta \), the conditional bias could be very large in absolute value.
3.3 Conditional bias reduction estimate

Denoted by \( \hat{\theta}_3 \), CBRE is obtained by subtracting an estimate of the conditional bias \( b(\hat{\theta}|\eta) \) from the MLE \( \hat{\theta} \). More specifically \( \hat{\theta}_3 \) is the solution to the following equation (in \( \theta \)):

\[
\theta = \hat{\theta} - b(\hat{\theta}|\eta) \tag{3.12}
\]

The Whitehead’s bias reduced estimator \( \hat{\theta} \) is obtained by solving the equation (in \( \theta \))

\[
\theta = \hat{\theta} - b(\hat{\theta}) \tag{3.13}
\]

which ignores the stopping time. However the marginal bias \( b(\hat{\theta}) \) involves the conditional bias not only at the stopping time \( t_\eta \) but also those after which are not so relevant. Intuitively, the equation (3.12) makes more sense than (3.13).

By equation (3.12),

\[
\hat{\theta}_3 = \hat{\theta} - b_{\hat{\theta}_3}(\hat{\theta}|\eta) = \hat{\theta} - (E_{\hat{\theta}_3}(\hat{\theta}|\eta) - \hat{\theta}_3)
\]

\[
\iff E_{\hat{\theta}_3}(\hat{\theta}|\eta) = \hat{\theta}
\]

\[
\iff E_{\hat{\theta}_3}(S_\eta|\eta) = s_\eta
\]

So the equations (3.4) and (3.12) are also equivalent. Therefore,

\[
\hat{\theta}_1 \equiv \hat{\theta}_2 \equiv \hat{\theta}_3
\]

which means the three conditional estimators MCLE, CME and CBRE are identical. Therefore, we’ll use one common notation \( \hat{\theta} \) to represent them and just need to consider the one most convenient
for us. The form of the CME turns out to be the easiest to use in an extensive simulation study shown later in Section 4.

3.4 Modification of MCLE and preliminary results

A preliminary simulation study is performed in the same scenario as the first example where a symmetric 2-sided O'Brien-Fleming type group sequential boundary is used with five interim looks at time 0.2, 0.4, 0.6, 0.8 and 1.0. The overall level $\alpha$ is 0.05 and true $\theta$ is 3 which corresponds to the power of the test 83.9%. Table 3.1 lists the results of the estimated bias, variance and mean squared error of the MLE $\hat{\theta}$, Whitehead’s bias reduction estimate $\tilde{\theta}$ and the proposed conditional estimator $\hat{\theta}$ at each stopping stage as well as in overall based on 100,000 simulations. The stopping probabilities at all interim stages are also listed in the table. The reason for choosing 100,000 to be simulation times is that in this scenario the probability of stopping very early ($\eta = 1$) is very low, only about 0.1 percent. By simulating 100,000 times we can obtain around 120 random samples at the first stopping stage and therefore the estimated bias, standard deviation and MSE of different estimators will not be too unreliable.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>overall</th>
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<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>7.41</td>
<td>2.63</td>
<td>0.99</td>
<td>0.08</td>
<td>-0.99</td>
<td>0.31</td>
</tr>
<tr>
<td>bias</td>
<td>6.97</td>
<td>2.33</td>
<td>0.65</td>
<td>-0.21</td>
<td>-1.17</td>
<td>0.04</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.41</td>
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<td>-0.13</td>
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</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>0.49</td>
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<td>0.53</td>
<td>0.41</td>
<td>0.66</td>
<td>1.31</td>
</tr>
<tr>
<td>Std</td>
<td>0.51</td>
<td>0.68</td>
<td>0.53</td>
<td>0.39</td>
<td>0.58</td>
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<td>2.11</td>
<td>1.57</td>
<td>1.97</td>
</tr>
<tr>
<td>MSE</td>
<td>55.12</td>
<td>7.37</td>
<td>1.26</td>
<td>0.18</td>
<td>1.42</td>
<td>1.81</td>
</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>48.78</td>
<td>5.87</td>
<td>0.71</td>
<td>0.19</td>
<td>1.71</td>
<td>1.56</td>
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<td>3.90</td>
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<tr>
<td>Prob($\eta$)</td>
<td>0.001</td>
<td>0.119</td>
<td>0.296</td>
<td>0.262</td>
<td>0.323</td>
<td></td>
</tr>
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Table 3.1: Table of the bias, standard deviation and MSE of MLE, Whitehead’s bias adjusted estimator and MCLE of $\theta$ when a 2-sided OBF type boundary is used with $T=(.2,.4,.6,.8,1)$ and $\theta = 3$ (power≈84%)

This table shows that in this case the conditional estimator $\hat{\theta}$ almost always results in smaller
bias than both the MLE $\hat{\theta}$ and the Whitehead's bias reduction estimator $\tilde{\theta}$ in all five interim stages and even in overall. It performs extremely well in term of the bias and the mean squared error when the trial is stopped at early stages. The biases of $\hat{\theta}$ and $\tilde{\theta}$ are most different from each other at the two most extreme stopping stages (1 and 5) and move closer when the stopping time is between them. We also observe a significant increase in standard deviation of the $\hat{\theta}$ at every stage. At early stopping stages because of the great reduction in bias, the MSE is still reduced substantially. However, at late stages, the MSE of the $\hat{\theta}$ is indeed the largest among the three. A reason of this might be that the MCLE takes too much into account the possibility of large values of $\theta$ which is not likely to be the case when the boundary is crossed late. Here then comes an issue of balancing between the bias and variance in point estimation. Looking for alternative estimators with both small bias and variance, two modified version of the $\tilde{\theta}$ are proposed which are hybrid forms of the conditional and unconditional MLE. The first one, denoted by $\tilde{\theta}'$, takes the value of $\hat{\theta}$ or $\tilde{\theta}$, whichever is closer to zero. By doing that, we don't inflate the MLE $\hat{\theta}$ but do shrink it if necessary. Specifically,

$$
\tilde{\theta}' = \begin{cases} 
\hat{\theta}, & \text{when } |\hat{\theta}| > |\tilde{\theta}| \\
\tilde{\theta}, & \text{otherwise}
\end{cases}
$$

Without loss of generality, we can assume $S_\eta$ is positive or equivalently the MLE $\hat{\theta} > 0$. First, let $\theta_{\eta m}$ be the solution to the equation (in $\theta$): $E_\theta[S_\eta|\eta] = c_\eta^*$ when $\eta < M$. The $\theta_{\eta m}$ is called a minimal estimation point (MEP). By the monotonicity of the conditional expectation, we know the MEP is unique for every $\eta < M$ and it is the smallest possible MCLE of the $\theta$ at that interim stage. The $\theta_{\eta m}$ is defined as 0 for $\eta = M$ because at the last interim the critical value $c_M^*$ is no longer involved in continuation criterion and thus the observed $\hat{\theta}$ has no lower limit. Second, let $\theta_{\eta c}$ ($> \theta_{\eta m}$) satisfy $E_{\theta_{\eta c}}[\hat{\theta}|\eta] = \theta_{\eta c}$ if it exists for $\eta < M$. The $\theta_{\eta c}$ is called an equilibrium estimation.
point (EEP) in the sense that if \( \theta = \theta_{\eta_e} \) the conditional expectation of \( \hat{\theta} \) at the stopping stage is the same as \( \theta_{\eta_e} \). It's easy to see that the EEP \( \theta_{\eta_e} \) should be \( +\infty \) for the first interim because as shown before \( E_{\theta}[\hat{\theta} \mid \eta = 1] > \theta \) for any \( \theta > 0 \). So the definition of \( \hat{\theta}' \) can be rewritten as

\[
\hat{\theta}' = \begin{cases} 
\hat{\theta}, & \text{if } \eta = M \text{ or } \hat{\theta} > \theta_{\eta_e} \\
\hat{\theta}, & \text{if } \eta < M \text{ and } \hat{\theta} < \theta_{\eta_e}
\end{cases}
\]

If replacing the conditional expectation curve between the MEP and the EEP by a straight line segment and using that as an approximate conditional expection, the other modified estimator \( \hat{\theta}'' \) is defined as

\[
\hat{\theta}'' = \begin{cases} 
\hat{\theta}, & \text{if } \eta = M \text{ or } \hat{\theta} > \theta_{\eta_e} \\
\theta_{\eta_e} - \frac{(\theta_{\eta_e} - \hat{\theta})(\theta_{\eta_e} - \theta_{\eta_e})}{\theta_{\eta_e} - c^*_{\eta} / t_\eta}, & \text{if } 1 < \eta < M \text{ and } \hat{\theta} < \theta_{\eta_e} \\
\hat{\theta}, & \text{if } \eta = 1
\end{cases}
\]

The definition of the \( \hat{\theta}' \) and \( \hat{\theta}'' \) can be easily extended to the case of negative \( \hat{\theta} \). Both of them adjust the \( \hat{\theta} \) but differently when it falls between the lower limit \( c^*_{\eta} / t_\eta \) and \( \theta_{\eta_e} \), where the \( \hat{\theta} \) tends to overestimate the true \( \theta \). However, they do not inflate the \( \hat{\theta} \) as the \( \hat{\theta} \) does at other occasions when it tends to underestimate the true \( \theta \).

4 More Simulation Results

Table 4.1 shows the results from the same 100,000-run simulation study of the last example in which the true \( \theta \) is 3 with approximate power of 84%. Both \( \hat{\theta}' \) and \( \hat{\theta}'' \) have much smaller standard deviation than \( \hat{\theta} \) after the first stage and the biases are not changed significantly. The mean squared errors (MSE) are consequently much reduced from the \( \hat{\theta} \). In fact, the MSE of the second modified
estimator $\hat{\theta}''$ is even better than the Whitehead's unconditional adjusted estimate $\bar{\theta}$.

<table>
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<tr>
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<th>1</th>
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<td>-0.16</td>
<td>-0.13</td>
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<td>-0.02</td>
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<td>$\tilde{\theta}'$</td>
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<td>0.66</td>
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<td>1.09</td>
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<td>0.296</td>
<td>0.262</td>
<td>0.323</td>
<td></td>
</tr>
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</table>

Table 4.1: Table of the bias, standard deviation and MSE of different estimators of $\theta$ when an OBF type boundary is used with $T=(.2,.4,.6,.8,1)$ and $\theta=3$ (power=83.9%)

A more extensive simulation study is performed to further evaluate and compare those different estimators. Four scenarios of the timing of interim analyses are considered, which are

1. five equally spaced analyses at time .2, .4, .6, .8 and 1.0.

2. four early pattern analyses at time .1, .3, .5 and 1.0.

3. four late pattern analyses at time .5, .7, .9 and 1.0.

4. three equally spaced analyses at time .33, .67 and 1.0.

For each of the above analysis pattern, three kinds of sequential boundaries are used for monitoring the Brownian motion. They correspond to a more conservative O’Brien-Fleming type, a more aggressive Pocock type and a moderate uniform type boundaries. All of them are symmetric two-sided so we don’t have to test using negative $\bar{\theta}$ because the result will be the same as using its absolute value. A Fortran program written by David Reboussion [13] is used to numerically calculate
the critical values for each interim analysis. The alpha spending functions used to generate the three types of boundaries are

1. O'Brien-Fleming type: \( \alpha_1(t) = 2 - 2\Phi(\frac{Z_{\alpha/2}}{\sqrt{t}}) \).

2. Pocock type: \( \alpha_2(t) = \alpha \cdot \ln(1 + (e - 1)t) \).

3. Uniform type: \( \alpha_3(t) = \alpha \cdot t \).

The \( \alpha \) is set at 0.025 for both the positive and negative trend so that the overall type I error is 0.05.

In addition, for each combination of a boundary type and an analysis pattern described above, Brownian motion processes \( S_\theta(t) \) with different drift parameter \( \theta \) are simulated under the true \( \theta \) values of 1, 3 and 5. When the parameter \( \theta = 3 \), the powers of those tests are between 77.6% and 84.9% which represent trials with moderate power. When the \( \theta = 1 \), the powers are between 13.6% and 16.7% which represent under-powered trials. When the \( \theta = 5 \), the powers are between 99.7% and 99.9% which represent over-powered trials. In each of the thirty six scenarios 100,000 random processes are simulated as before. The estimated bias, variance and mean squared error from the simulated results for those five estimators at each possible stopping stage are listed in the Appendix A.4 (Table A.1–A.36).

From those tables, we can see that when the overall bias is the only concern, the Whitehead’s bias adjusted estimator \( \hat{\theta} \) is the least biased in most case. However, as discussed before, the small overall bias may obscure the large conditional biases because they can offset each other at different stopping times. Similar to what observed in Table 4.1 before, the difference between the conditional biases of \( \hat{\theta} \) and the MLE \( \hat{\theta} \) are most significant at the first stopping stage. As the stopping time is postponed, the two conditional biases become closer, kind of cross each other in the middle and depart again at the end. Generally \( \hat{\theta} \) is less conditionally biased than the usual MLE \( \hat{\theta} \) and
the Whitehead’s bias adjusted estimator \( \hat{\theta} \) but with significantly larger variance. As a result, the conditional MLE \( \hat{\theta} \) is often less satisfactory than the \( \hat{\theta} \) and \( \bar{\theta} \) in term of the mean squared error.

The two hybrid estimators \( \hat{\theta}' \) and \( \hat{\theta}'' \) take advantage of both the smaller conditional bias of \( \hat{\theta} \) and smaller variance of \( \bar{\theta} \). They shrink the \( \hat{\theta} \) when it crosses the boundary only by a small amount, which is more likely to happen by chance. On the other hand, they are identical to \( \bar{\theta} \) when it crosses the boundary by a relatively large margin, which is more likely to happen due to the real effect of the drift parameter. Both of them, especially \( \hat{\theta}'' \), reduce the variance from \( \hat{\theta} \) in most cases. Compared with \( \hat{\theta} \), the \( \hat{\theta}'' \) has a much smaller MSE when it’s stopped at the first interim except in some extremely high power case (i.e. \( \theta = 5 \)) or in the cases that the critical value at the first interim analysis is small. At other stopping stages, the \( \hat{\theta}'' \) and \( \bar{\theta} \) perform similarly in term of bias and MSE. When the trial is under-powered, e.g. \( \theta = 1 \), the \( \hat{\theta}' \) has the least bias and MSE in most cases because it tends to adjust the \( \hat{\theta} \) most aggressively.

When comparing different boundary types, it is observed that generally the more conservative the boundary, which can be measured by the convexity of the corresponding alpha spending function, the better the estimators are in term of the overall MSE. The MSE is usually the smallest under the O’Brien-Fleming type boundary and the largest under the Pocock type boundary. That means more conservatism in spending the overall \( \alpha \) will not only increase the power of the sequential test but also improve the estimation.

5 Discussions

In a group sequential design, the unconditional version of \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are the MLE \( \hat{\theta} \) and Whitehead’s bias reduced estimator \( \bar{\theta} \) respectively, which are two different values. The unconditional version of equation (3.3), i.e. \( E_{\theta}[(n, S_n)] = (n, s_n) \), usually does not have a solution because there’s only one parameter but two constraints in the equation. However, when conditioning on the stopping time,
those three estimators, equation (3.1), (3.4) and (3.12), become identical.

From definition, the conditional estimator $\hat{\theta}$ depends only on the information at the stopping interim stage and the stages before. A parameter estimator is called future independent if it does not depend on the number, timing and boundary values of those interim stages planned but not used after the experiment has been early stopped at any time. This is a very desirable feature because those future but yet happening interim analyses have nothing to do with the trial going on so far and thus should not affect the estimation at current stage. Furthermore, in flexible trials the future interim analyses may be uncertain, e.g. an additional interim could be added or the data might be cumulated faster or slower than planned. The possible alternation of either the number or the time makes the dependence upon the future information very difficult and unnatural to explain. Both the $\hat{\theta}$ and the MLE $\hat{\theta}$ are future independent but the Whitehead’s bias reduction estimator $\hat{\theta}$ is not. Also because of the future independence property, $\hat{\theta}$ is computationally easier to calculate than $\hat{\theta}$ when very early stopping happens. Since the two modified estimators $\hat{\theta}'$ and $\hat{\theta}''$ are just the functions of the MCLE $\hat{\theta}$ and the MLE $\hat{\theta}$, they are both independent of future information as well.

Emerson (1993) proposed an unbiased estimator of $\theta$ which is in the form of $E[\hat{\theta}_1 | \eta, S_\eta]$. It was proposed because the author speculated that the doublet $(\eta, S_\eta)$ is a complete sufficient statistic for the parameter $\theta$ so $E[\hat{\theta}_1 | \eta, S_\eta]$ would be a UMVUE. However, Liu and Hall (1999) prove that the statistic is sufficient but not complete which undermines the justification of that estimator. Liu and Hall did show that it is uniformly minimum variance unbiased among a class of estimators so called “truncation-adaptable” which means that the estimator won’t be affected by removing any continuation region, i.e. changing any $A_k$ to an empty set. Again, the unbiasedness here still refers to unconditional bias. The conditional bias of Emerson’s estimator will be studied in the future. However, by knowing it doesn’t even adjust from $\hat{\theta}$ when the trial is stopped at the first stage (i.e. $\eta = 1$), we suspect it to have the same problem of prominent conditional biases in spite of the
zero overall bias. Furthermore, the “truncation-adaptable” in Liu and Hall (1999) only describes a very small class of estimators and it’s unintuitive to require an estimator take the same form under different stopping rules. By knowing the reason of stopping a sequential trial, whether because the observed statistic exceeds the current critical bound or because it’s the scheduled final interim, we can and should be able to use different form of estimators.

To our knowledge, the conditional estimator with its modifications has not been much explored before. The new method takes the advantage of the discrete stopping rule and the overshoot of the test statistic at the stopping time. In many cases they provide us a better estimator to the unknown parameter than the available methods. Although not always out-perform others, they certainly give us a useful option to chooses from.

References


A Appendix

A.1

Proof of Theorem 1

Proof: Because the $A_1$ is symmetric, we can assume $\theta > 0$ without loss of generality. Let $\phi(x)$ denote the standard normal density. Then the conditional distribution of $S_1|\tau = 1$ under the drift parameter $\theta$ has the density $f_\theta(s) \propto \frac{1}{\sqrt{t_1}} \phi((s - \theta t_1)/\sqrt{t_1})I_{A_1^c}(s)$ where $I$ is an indicator function. So

$$E_\theta(S_1|\eta = 1) = \int s \cdot f_\theta(s) ds$$

$$= \int s \cdot f_\theta(s) I_{A_1^c \cap (-\varepsilon_1, \varepsilon_1^2 + 2\theta_1^2)}(s) ds + \int s \cdot f_\theta(s) I_{(\varepsilon_1, \varepsilon_1^2 + 2\theta_1^2)}(s) ds$$

The first integral above equals $\theta t_1$ and the second integral is greater than zero. So $E_\theta(S_1|\eta = 1) > \theta t_1$ and therefore

$$E_\theta(\hat{\theta}|\eta = 1) = E_\theta(S_1|\eta = 1)/t_1 > \theta.$$ 

As $\theta \to +\infty$, the density of $S_1 - \theta t_1|\eta = 1 \propto \frac{1}{\sqrt{t_1}} \phi(s/\sqrt{t_1})I_{(A_1 - \theta t_1)}(s)$ and it converges to $\frac{1}{\sqrt{t_1}} \phi(s/\sqrt{t_1})$ uniformly. Therefore

$$b(\hat{\theta}|\eta = 1) = \frac{1}{t_1} E_\theta(S_1 - \theta t_1|\eta = 1) = \int t_1^{-3/2} \phi(s/\sqrt{t_1}) ds = 0.$$ 

as $\theta \to \infty$. □
A.2

Proof of Lemma 2.1:

Proof: Similar to Armitage, P, etc. (1969), $f_k(s)$ can be recursively defined by

$$f_1(s) = \phi_t(s), \quad \text{and}$$

$$f_{k+1}(s) = \int_{b_k^*}^{a_k^*} f_k(x) \cdot \phi_{t_k-1-t_k}(s-x) dx \quad \text{for } k > 1.$$ 

where $\phi_t(s) = \frac{1}{\sqrt{2\pi}} \phi(s/\sqrt{t})$, the normal density with mean zero and variance $t$.

Because the centers $a_k = (b_k^* + c_k^*)/2 < 0$, so $0 > b_k^* < -c_k^$, $k = 1, \ldots, M - 1$.

By induction method, when $M = 2$, then $0 > b_1^* < -c_1^*$, so $[0, \max(0, c_1^*)] \subset [0, -b_1^*]$. We know that $\phi_{t}(s) = \phi_{t}(-s)$. For any $s < 0$, in the inequality (2.3),

$$LHS = \phi_{t_1}(s) \cdot I_{(b_1^*, \min(c_1^*, 0))}(s) \geq \phi_{t_1}(s) \cdot I_{(0, \max(0, c_1^*))}(-s) = RHS$$

and $LHS > RHS$ when $s \in (b_1^*, -c_1^*)$.

Suppose that the conclusion holds for $M = n$, i.e. we have

$$f_{n-1}(s) \cdot I_{A_{n-1} \cap (-\infty, 0)}(s) \geq f_{n-1}(-s) \cdot I_{A_{n-1} \cap (0, \infty)}(-s)$$

for $s < 0$ and the strict inequality holds in a non-degenerated interval. Now we increase $M$ to be $n+1$, 

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then because \(-b_{n-1}^* > c_{n-1}^*,\) for any \(s \in (b_{n-1}^*, 0) \cap A_n\)

\[
f_n(s) - f_n(-s)
= \int_{b_{n-1}^*}^{c_{n-1}^*} f_{n-1}(x) \cdot \phi_{t_{n-1} - t_{n-1}}(s - x) dx - \int_{b_{n-1}^*}^{c_{n-1}^*} f_{n-1}(x) \cdot \phi_{t_{n-1} - t_{n-1}}(-s - x) dx
= \int_{b_{n-1}^*}^{0} f_{n-1}(x) \cdot I_{A_{n-1} \cap (-\infty, 0)}(x) \cdot [\phi_{t_{n-1} - t_{n-1}}(s - x) - \phi_{t_{n-1} - t_{n-1}}(-s - x)] dx
+ \int_{0}^{c_{n-1}^*} f_{n-1}(x) \cdot I_{A_{n-1} \cap (0, \infty)}(x) \cdot [\phi_{t_{n-1} - t_{n-1}}(s - x) - \phi_{t_{n-1} - t_{n-1}}(-s - x)] dx
= \int_{b_{n-1}^*}^{0} f_{n-1}(x) \cdot I_{A_{n-1} \cap (-\infty, 0)}(x) \cdot [\phi_{t_{n-1} - t_{n-1}}(s - x) - \phi_{t_{n-1} - t_{n-1}}(-s - x)] dx
+ \int_{0}^{c_{n-1}^*} f_{n-1}(-x) \cdot I_{A_{n-1} \cap (0, \infty)}(-x) \cdot [\phi_{t_{n-1} - t_{n-1}}(s + x) - \phi_{t_{n-1} - t_{n-1}}(-s + x)] dx
= \int_{b_{n-1}^*}^{0} [f_{n-1}(x) \cdot I_{A_{n-1} \cap (0, \infty)}(x) - f_{n-1}(-x) \cdot I_{A_{n-1} \cap (0, \infty)}(-x)] \cdot [\phi_{t_{n-1} - t_{n-1}}(s - x) - \phi_{t_{n-1} - t_{n-1}}(s + x)] dx
\]

because \(\phi_{t_{n-1} - t_{n-1}}(s - x) - \phi_{t_{n-1} - t_{n-1}}(s + x) > 0\) for \(s \in (b_{n-1}^*, 0), x < 0\) and by the induction assumption:

\[f_{n-1}(s) \cdot I_{A_{n-1} \cap (-\infty, 0)}(s) \geq f_{n-1}(-s) \cdot I_{A_{n-1} \cap (0, \infty)}(-s)\]

with the strict inequality held in a non-degenerated interval, it follows that

\[f_n(s) \cdot I_{A_n \cap (-\infty, 0)}(s) > f_n(-s) \cdot I_{A_n \cap (0, \infty)}(-s) \geq f_n(-s) \cdot I_{A_n \cap (0, \infty)}(-s)\]

□
A.3

Proof of Lemma 3.1:

Proof: Let \( f_{k,\theta}(s_k) \) denote the density of \( S_k \) under \( \theta \) which is defined similar to the one by Armitage, P et al (1969), i.e.:

\[
f_{1,\theta}(s_1) = p_{\theta,1}(x_1)
\]

\[
f_{k+1,\theta}(s_{k+1}) = \int_{A_2^*} p_{\theta, k+1}(s_{k+1} - s_k) f_{k,\theta}(s_k) ds_k, \quad k = 1, 2, \ldots
\]

First, we'll show for any integer \( n \),

\[
\frac{\partial}{\partial \theta} f_{n,\theta}(s_n) = \sum_{i=1}^{n} [T_i(x_i) \cdot \lambda'_i(\theta) - B'_i(\theta)] f_{n,\theta}(s_n) \tag{A.1}
\]

By induction, for \( n = 1 \), since \( f_{1,\theta}(s_1) = p_{\theta,1}(x_1) = h_1(x) \exp(\lambda_1(\theta) \cdot T_1(x) - B_1(\theta)) \), it's easy to see the equation (A.1) holds. Assume (A.1) holds for \( n = k \). Let \( n \) increase by 1. Then, because of the property of exponential family,

\[
\frac{\partial}{\partial \theta} f_{k+1,\theta}(s_{k+1}) = \int_{A_2^*} \frac{\partial}{\partial \theta} \{ p_{\theta, k+1}(s_{k+1} - s_k) f_{k,\theta}(s_k) \} ds_k
\]

\[
= \int_{A_2^*} \left[ T_{k+1}(s_{k+1} - s_k) \cdot \lambda'_{k+1}(\theta) - B'_{k+1}(\theta) \right] p_{\theta, k+1}(s_{k+1} - s_k) f_{k,\theta}(s_k) ds_k
\]

\[
+ p_{\theta, k+1}(s_{k+1} - s_k) \frac{\partial}{\partial \theta} f_{k,\theta}(s_k) ds_k
\]

\[
= \int_{A_2^*} \left[ T_{k+1}(s_{k+1} - s_k) \cdot \lambda'_{k+1}(\theta) - B'_{k+1}(\theta) \right] p_{\theta, k+1}(s_{k+1} - s_k) f_{k,\theta}(s_k) ds_k
\]

\[
+ p_{\theta, k+1}(s_{k+1} - s_k) \sum_{i=1}^{k} [T_i(x_i) \cdot \lambda'_i(\theta) - B'_i(\theta)] f_{k,\theta}(s_k) ds_k
\]

\[
= \sum_{i=1}^{k+1} [T_i(x_i) \cdot \lambda'_i(\theta) - B'_i(\theta)] f_{k+1,\theta}(s_{k+1})
\]

\[
= \sum_{i=1}^{k+1} [T_i(x_i) \cdot \lambda'_i(\theta) - B'_i(\theta)] f_{k+1,\theta}(s_{k+1})
\]
So the equation (A.1) holds for any $n$. Also

$$E_\theta[g(S_\eta, t_\eta)] = \sum_{n=1}^{M} E_\theta[g(S_\eta, t_\eta) | \eta = n] P(\eta = n)$$

$$= \sum_{n=1}^{M} \int_{A_\eta^n} g(s_n, t_n) \frac{f_{n, \theta}(s_n)}{P(\eta = n)} ds_n P(\eta = n)$$

$$= \sum_{n=1}^{M} \int_{A_\eta^n} g(s_n, t_n) f_{n, \theta}(s_n) ds_n$$

The result follows from by taking derivative w.r.t $\theta$ in the both sides of the above equation, exchanging summation and derivative signs and then plugging equation (A.1) in. \hfill \Box

A.4 Tables
<table>
<thead>
<tr>
<th>$n = \cdot$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
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<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>9.84</td>
<td>4.28</td>
<td>2.64</td>
<td>1.82</td>
<td>-0.16</td>
<td>0.07</td>
</tr>
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<td>0.16</td>
<td>0.22</td>
</tr>
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<td>0.54</td>
<td>-0.16</td>
<td>-0.09</td>
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<td>0.83</td>
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<td>0.77</td>
<td>1.31</td>
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<td>0.65</td>
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<td>0.90</td>
<td>0.56</td>
<td>0.77</td>
<td>1.16</td>
</tr>
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<td>0.70</td>
<td>1.10</td>
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<td>1.89</td>
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<td>1.17</td>
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<tr>
<td>Prob($\eta$)</td>
<td>$4 \times 10^{-6}$</td>
<td>0.007</td>
<td>0.033</td>
<td>0.056</td>
<td>0.904</td>
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</tr>
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</table>

Table A.1: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.2, .4, .6, .8, 1)$ and $\theta = 1$ (power$\approx$16.5%)  
† based on 100,000 simulations

<table>
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<tr>
<th>$n = \cdot$</th>
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<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
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<td>0.08</td>
<td>-0.99</td>
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<td>-0.03</td>
</tr>
<tr>
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<td>-0.47</td>
<td>-0.81</td>
<td>-0.99</td>
<td>-0.70</td>
</tr>
<tr>
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<td>-0.99</td>
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</tr>
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<td>1.70</td>
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<tr>
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<td>0.44</td>
<td>0.28</td>
<td>0.15</td>
<td>0.34</td>
<td>1.55</td>
</tr>
<tr>
<td>$\hat{\delta}'$</td>
<td>11.30</td>
<td>4.56</td>
<td>4.39</td>
<td>4.42</td>
<td>2.44</td>
<td>3.85</td>
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<td>4.47</td>
<td>2.70</td>
<td>1.48</td>
<td>0.42</td>
<td>1.94</td>
</tr>
<tr>
<td>$\hat{\delta}''$</td>
<td>11.30</td>
<td>0.73</td>
<td>0.56</td>
<td>0.43</td>
<td>0.42</td>
<td>1.50</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
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<td>1.24</td>
<td>0.18</td>
<td>1.41</td>
<td>1.80</td>
</tr>
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<td>49.8</td>
<td>5.84</td>
<td>0.70</td>
<td>0.19</td>
<td>1.70</td>
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<td>4.43</td>
<td>4.45</td>
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<tr>
<td>$\hat{\delta}''$</td>
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<td>4.54</td>
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<td>1.07</td>
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<tr>
<td>Prob($\eta$)</td>
<td>0.001</td>
<td>0.118</td>
<td>0.296</td>
<td>0.262</td>
<td>0.322</td>
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Table A.2: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.2, .4, .6, .8, 1)$ and $\theta = 3$ (power$\approx$83.9%)  
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$n = 1$</th>
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<th>4</th>
<th>5</th>
<th>overall</th>
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<tbody>
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<td>1.12</td>
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<td>-1.56</td>
<td>-2.30</td>
</tr>
<tr>
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<td>5.18</td>
<td>0.82</td>
<td>-0.89</td>
<td>-1.87</td>
<td>-2.55</td>
</tr>
<tr>
<td>bias $\hat{\theta}$</td>
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<td>-0.76</td>
<td>-0.48</td>
<td>-0.41</td>
<td>0.24</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
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<td>-0.83</td>
<td>-1.31</td>
<td>-1.99</td>
<td>-2.30</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
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<td>0.74</td>
<td>-0.69</td>
<td>-1.66</td>
<td>-2.30</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
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<td>0.86</td>
<td>0.49</td>
<td>0.28</td>
<td>0.30</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
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<td>0.86</td>
<td>0.50</td>
<td>0.26</td>
<td>0.25</td>
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<td>6.72</td>
<td>6.24</td>
<td>6.25</td>
<td>3.37</td>
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<td>6.28</td>
<td>2.73</td>
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<td>0.30</td>
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<td>1.29</td>
<td>3.74</td>
<td>6.75</td>
</tr>
<tr>
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<td>7.33</td>
<td>6.47</td>
<td>6.42</td>
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<tr>
<td>$\hat{\theta}'$</td>
<td>13.7</td>
<td>6.98</td>
<td>4.44</td>
<td>5.27</td>
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<tr>
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<td>1.91</td>
<td>1.22</td>
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<td>0.520</td>
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Table A.3: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.2, .4, .6, .8, 1)$ and $\theta = 5$ (power $\approx 99.9\%$)

† based on 100,000 simulations

<table>
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<tr>
<th>$\eta = 1$</th>
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<th>5</th>
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<td>2.37</td>
<td>1.73</td>
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<td>-0.31</td>
</tr>
<tr>
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<td>0.69</td>
<td>0.58</td>
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<td>2.82</td>
<td>2.10</td>
<td>1.57</td>
<td>-0.15</td>
</tr>
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<td>0.92</td>
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<tr>
<td>$\hat{\theta}'$</td>
<td>6.5</td>
<td>3.5</td>
<td>2.24</td>
<td>1.65</td>
<td>0.81</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>6.5</td>
<td>10.3</td>
<td>5.35</td>
<td>3.03</td>
<td>0.81</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.025</td>
<td>0.028</td>
<td>0.027</td>
<td>0.027</td>
<td>0.003</td>
</tr>
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</table>

Table A.4: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.2, .4, .6, .8, 1)$ and $\theta = 1$ (power $\approx 13.6\%$)

† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$q$ =</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>3.56</td>
<td>1.60</td>
<td>0.67</td>
<td>0.15</td>
<td>-0.93</td>
<td>0.63</td>
</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>3.06</td>
<td>0.90</td>
<td>0.02</td>
<td>-0.42</td>
<td>-1.29</td>
<td>0.10</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>0.22</td>
<td>-0.06</td>
<td>-0.16</td>
<td>-0.14</td>
<td>0.26</td>
<td>0.05</td>
</tr>
<tr>
<td>$\hat{\theta}$''</td>
<td>0.22</td>
<td>-0.25</td>
<td>-0.63</td>
<td>-0.85</td>
<td>-0.93</td>
<td>-0.57</td>
</tr>
<tr>
<td>$\hat{\theta}$'''</td>
<td>0.22</td>
<td>1.38</td>
<td>0.48</td>
<td>-0.10</td>
<td>-0.93</td>
<td>0.06</td>
</tr>
</tbody>
</table>

| $\hat{\theta}$ | 1.07  | 0.41  | 0.21  | 0.13  | 0.45  | 2.66    |
| $\hat{\theta}$' | 1.37  | 0.45  | 0.17  | 0.09  | 0.29  | 2.44    |
| $\hat{\theta}$'' | 5.66  | 4.92  | 4.84  | 5.00  | 2.29  | 4.17    |
| $\hat{\theta}$''' | 5.66  | 3.55  | 2.19  | 1.56  | 0.45  | 2.41    |
| $\hat{\theta}$'''' | 5.66  | 0.61  | 0.38  | 0.37  | 0.45  | 1.88    |

| $\hat{\theta}$ | 13.8  | 2.95  | 0.66  | 0.15  | 1.31  | 3.05    |
| $\hat{\theta}$' | 10.7  | 1.26  | 0.17  | 0.27  | 1.94  | 2.45    |
| $\hat{\theta}$'' | 5.7   | 4.92  | 4.87  | 5.02  | 2.36  | 4.17    |
| $\hat{\theta}$''' | 5.7   | 3.61  | 2.59  | 2.28  | 1.31  | 2.74    |
| $\hat{\theta}$'''' | 5.7   | 2.53  | 0.61  | 0.38  | 1.31  | 1.88    |

| Prob($\eta$) | 0.138 | 0.192 | 0.185 | 0.148 | 0.337 |         |

Table A.5: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.2,.4,.6,.8,1)$ and $\theta = 3$ (power $\approx 77.6\%$)

† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta$ =</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>2.08</td>
<td>0.64</td>
<td>-0.98</td>
<td>-1.59</td>
<td>-2.20</td>
<td>0.64</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.65</td>
<td>-0.63</td>
<td>-1.65</td>
<td>-2.21</td>
<td>-2.70</td>
<td>0.08</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>-0.61</td>
<td>-0.62</td>
<td>-0.55</td>
<td>-0.61</td>
<td>0.19</td>
<td>-0.59</td>
</tr>
<tr>
<td>$\hat{\theta}$''</td>
<td>-0.61</td>
<td>-1.13</td>
<td>-1.70</td>
<td>-2.14</td>
<td>-2.20</td>
<td>-1.06</td>
</tr>
<tr>
<td>$\hat{\theta}$'''</td>
<td>-0.61</td>
<td>-0.09</td>
<td>-1.07</td>
<td>-1.72</td>
<td>-2.20</td>
<td>-0.57</td>
</tr>
</tbody>
</table>

| $\hat{\theta}$ | 1.6   | 0.72  | 0.36  | 0.22  | 0.27  | 2.76    |
| $\hat{\theta}$' | 2.1   | 0.84  | 0.33  | 0.16  | 0.17  | 3.28    |
| $\hat{\theta}$'' | 8.3   | 7.16  | 6.95  | 6.95  | 2.77  | 7.52    |
| $\hat{\theta}$''' | 8.3   | 4.24  | 2.24  | 1.48  | 0.27  | 5.67    |
| $\hat{\theta}$'''' | 8.3   | 0.95  | 0.53  | 0.42  | 0.27  | 4.16    |

| $\hat{\theta}$ | 3.92  | 0.72  | 1.32  | 2.75  | 5.11  | 3.17    |
| $\hat{\theta}$' | 4.79  | 1.23  | 3.06  | 5.02  | 7.48  | 3.28    |
| $\hat{\theta}$'' | 8.62  | 7.55  | 7.26  | 7.32  | 2.81  | 7.87    |
| $\hat{\theta}$''' | 8.62  | 5.51  | 5.14  | 6.08  | 5.11  | 6.79    |
| $\hat{\theta}$'''' | 8.62  | 0.96  | 1.68  | 3.39  | 5.11  | 4.48    |

| Prob($\eta$) | 0.423 | 0.365 | 0.151 | 0.045 | 0.015 |         |

Table A.6: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.2,.4,.6,.8,1)$ and $\theta = 5$ (power $\approx 99.7\%$)

† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$\eta$ =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>4.69</td>
<td>3.22</td>
<td>2.40</td>
<td>1.88</td>
<td>-0.16</td>
<td>0.15</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>4.16</td>
<td>2.55</td>
<td>1.80</td>
<td>1.38</td>
<td>-0.31</td>
<td>-0.04</td>
</tr>
<tr>
<td>bias $\hat{\theta}'$</td>
<td>1.19</td>
<td>0.95</td>
<td>0.87</td>
<td>0.79</td>
<td>0.15</td>
<td>0.23</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>1.19</td>
<td>0.90</td>
<td>0.71</td>
<td>0.53</td>
<td>-0.16</td>
<td>-0.06</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.19</td>
<td>2.96</td>
<td>2.12</td>
<td>1.50</td>
<td>-0.16</td>
<td>0.06</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>11.2</td>
<td>2.34</td>
<td>0.87</td>
<td>0.37</td>
<td>0.78</td>
<td>1.92</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>11.4</td>
<td>2.34</td>
<td>0.84</td>
<td>0.34</td>
<td>0.64</td>
<td>1.57</td>
</tr>
<tr>
<td>Var $\hat{\theta}$</td>
<td>5.3</td>
<td>3.04</td>
<td>2.82</td>
<td>2.70</td>
<td>1.68</td>
<td>1.91</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>5.3</td>
<td>2.63</td>
<td>1.75</td>
<td>1.29</td>
<td>0.78</td>
<td>1.04</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>5.3</td>
<td>2.19</td>
<td>0.89</td>
<td>0.50</td>
<td>0.78</td>
<td>1.36</td>
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<tr>
<td>$\hat{\theta}$</td>
<td>33.2</td>
<td>12.7</td>
<td>6.61</td>
<td>3.89</td>
<td>0.81</td>
<td>1.94</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>28.7</td>
<td>8.9</td>
<td>4.09</td>
<td>2.25</td>
<td>0.73</td>
<td>1.57</td>
</tr>
<tr>
<td>MSE $\hat{\theta}$</td>
<td>6.8</td>
<td>3.9</td>
<td>3.58</td>
<td>3.32</td>
<td>1.70</td>
<td>1.96</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>6.8</td>
<td>3.4</td>
<td>2.26</td>
<td>1.48</td>
<td>0.81</td>
<td>1.04</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>6.8</td>
<td>10.9</td>
<td>5.40</td>
<td>2.76</td>
<td>0.81</td>
<td>1.36</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.017</td>
<td>0.024</td>
<td>0.030</td>
<td>0.034</td>
<td>0.894</td>
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</tr>
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</table>

Table A.7: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T = (2, 4, 6, 8, 1)$ and $\theta = 1$ (power$\approx 14.5\%$)
† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta$ =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>3.82</td>
<td>1.73</td>
<td>0.70</td>
<td>0.11</td>
<td>-0.96</td>
<td>0.59</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.32</td>
<td>1.07</td>
<td>0.69</td>
<td>-0.42</td>
<td>-1.28</td>
<td>0.09</td>
</tr>
<tr>
<td>bias $\hat{\theta}'$</td>
<td>0.21</td>
<td>-0.06</td>
<td>-0.15</td>
<td>-0.13</td>
<td>0.27</td>
<td>0.05</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>0.21</td>
<td>-0.23</td>
<td>-0.60</td>
<td>-0.84</td>
<td>-0.96</td>
<td>-0.60</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.0</td>
<td>0.41</td>
<td>0.22</td>
<td>0.14</td>
<td>0.43</td>
<td>2.61</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.2</td>
<td>0.44</td>
<td>0.18</td>
<td>0.10</td>
<td>0.29</td>
<td>2.35</td>
</tr>
<tr>
<td>Var $\hat{\theta}$</td>
<td>5.9</td>
<td>5.00</td>
<td>4.79</td>
<td>4.84</td>
<td>2.33</td>
<td>4.19</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>5.9</td>
<td>3.72</td>
<td>2.26</td>
<td>1.52</td>
<td>0.43</td>
<td>2.35</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>5.9</td>
<td>0.62</td>
<td>0.41</td>
<td>0.38</td>
<td>0.43</td>
<td>1.83</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>15.5</td>
<td>3.39</td>
<td>0.71</td>
<td>0.15</td>
<td>1.35</td>
<td>2.95</td>
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<tr>
<td>$\hat{\theta}$</td>
<td>12.3</td>
<td>1.57</td>
<td>0.19</td>
<td>0.27</td>
<td>1.93</td>
<td>2.36</td>
</tr>
<tr>
<td>MSE $\hat{\theta}$</td>
<td>5.9</td>
<td>5.01</td>
<td>4.81</td>
<td>4.86</td>
<td>2.40</td>
<td>4.19</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>5.9</td>
<td>3.77</td>
<td>2.62</td>
<td>2.22</td>
<td>1.35</td>
<td>2.71</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>5.9</td>
<td>2.89</td>
<td>0.66</td>
<td>0.40</td>
<td>1.35</td>
<td>1.84</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.110</td>
<td>0.187</td>
<td>0.203</td>
<td>0.172</td>
<td>0.328</td>
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</table>

Table A.8: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T = (2, 4, 6, 8, 1)$ and $\theta = 3$ (power$\approx 79.8\%$)
† based on 100,000 simulations
<table>
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<th>4</th>
<th>5</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>2.29</td>
<td>0.17</td>
<td>-0.94</td>
<td>-1.62</td>
<td>-2.28</td>
<td>0.64</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
<td>1.86</td>
<td>-0.48</td>
<td>-1.57</td>
<td>-2.18</td>
<td>-2.73</td>
<td>0.08</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime$</td>
<td>-0.64</td>
<td>-0.63</td>
<td>-0.57</td>
<td>-0.56</td>
<td>0.11</td>
<td>-0.61</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime\prime$</td>
<td>-0.64</td>
<td>-1.09</td>
<td>-1.67</td>
<td>-2.13</td>
<td>-2.28</td>
<td>-1.09</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime\prime\prime$</td>
<td>-0.64</td>
<td>-0.03</td>
<td>-1.04</td>
<td>-1.74</td>
<td>-2.28</td>
<td>-0.53</td>
</tr>
<tr>
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<td>1.5</td>
<td>0.72</td>
<td>0.39</td>
<td>0.23</td>
<td>0.30</td>
<td>2.81</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
<td>1.9</td>
<td>0.81</td>
<td>0.35</td>
<td>0.17</td>
<td>0.20</td>
<td>3.23</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
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<td>7.20</td>
<td>6.86</td>
<td>6.79</td>
<td>3.05</td>
<td>7.56</td>
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<td>8.5</td>
<td>4.45</td>
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<td>1.43</td>
<td>0.30</td>
<td>5.58</td>
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<td>8.5</td>
<td>0.96</td>
<td>0.56</td>
<td>0.43</td>
<td>0.30</td>
<td>3.94</td>
</tr>
<tr>
<td>MSE</td>
<td>6.72</td>
<td>0.75</td>
<td>1.26</td>
<td>2.84</td>
<td>5.48</td>
<td>3.21</td>
</tr>
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<td>$\hat{\delta}$</td>
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<td>1.04</td>
<td>2.82</td>
<td>4.94</td>
<td>7.64</td>
<td>3.23</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>8.95</td>
<td>7.59</td>
<td>7.18</td>
<td>7.10</td>
<td>3.06</td>
<td>7.93</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime$</td>
<td>8.95</td>
<td>5.63</td>
<td>5.10</td>
<td>5.97</td>
<td>5.48</td>
<td>6.77</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime\prime$</td>
<td>8.95</td>
<td>0.96</td>
<td>1.65</td>
<td>3.46</td>
<td>5.48</td>
<td>4.22</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.369</td>
<td>0.391</td>
<td>0.177</td>
<td>0.049</td>
<td>0.014</td>
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</tr>
</tbody>
</table>

Table A.9: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=\{2,4,6,8,1\}$ and $\theta = 5$ (power≈99.7%)
† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta$</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>6.73</td>
<td>3.56</td>
<td>-0.026</td>
<td>0.017</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
<td>NA</td>
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<td>3.21</td>
<td>-0.060</td>
<td>0.021</td>
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<tr>
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<td>NA</td>
<td>1.65</td>
<td>0.75</td>
<td>0.023</td>
<td>0.032</td>
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<td>NA</td>
<td>1.65</td>
<td>0.74</td>
<td>-0.026</td>
<td>-0.017</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime\prime$</td>
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<td>5.74</td>
<td>3.01</td>
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<td>0.011</td>
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<td>0.96</td>
<td>1.12</td>
</tr>
<tr>
<td>$\hat{\delta}$</td>
<td>NA</td>
<td>0.44</td>
<td>0.91</td>
<td>0.88</td>
<td>1.01</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>6.13</td>
<td>2.55</td>
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<td>1.12</td>
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<tr>
<td>$\hat{\theta}^\prime$</td>
<td>NA</td>
<td>6.13</td>
<td>2.42</td>
<td>0.96</td>
<td>0.98</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime\prime$</td>
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<td>0.54</td>
<td>1.05</td>
<td>0.96</td>
<td>1.07</td>
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<td>13.7</td>
<td>0.96</td>
<td>1.12</td>
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<td>41.8</td>
<td>11.2</td>
<td>0.88</td>
<td>1.01</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>8.9</td>
<td>3.1</td>
<td>1.10</td>
<td>1.12</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime$</td>
<td>NA</td>
<td>8.9</td>
<td>3.0</td>
<td>0.96</td>
<td>0.99</td>
</tr>
<tr>
<td>$\hat{\theta}^\prime\prime$</td>
<td>NA</td>
<td>33.5</td>
<td>10.1</td>
<td>0.96</td>
<td>1.07</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.000</td>
<td>0.0003</td>
<td>0.011</td>
<td>0.988</td>
<td></td>
</tr>
</tbody>
</table>

Table A.10: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=\{1,3,5,1\}$ and $\theta = 1$ (power≈16.7%)
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$\eta = \frac{1}{1}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>4.80</td>
<td>1.92</td>
<td>-0.25</td>
<td>0.211</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>4.50</td>
<td>1.56</td>
<td>-0.41</td>
<td>0.007</td>
</tr>
<tr>
<td>bias $\hat{\theta}$</td>
<td>NA</td>
<td>-0.05</td>
<td>-0.28</td>
<td>0.07</td>
<td>0.006</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>-0.05</td>
<td>-0.34</td>
<td>-0.25</td>
<td>-0.264</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>3.82</td>
<td>1.48</td>
<td>-0.25</td>
<td>0.119</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>0.33</td>
<td>0.36</td>
<td>0.79</td>
<td>1.65</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>0.38</td>
<td>0.35</td>
<td>0.64</td>
<td>1.39</td>
</tr>
<tr>
<td>Var $\hat{\theta}$</td>
<td>NA</td>
<td>6.39</td>
<td>4.39</td>
<td>1.30</td>
<td>1.95</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>6.39</td>
<td>2.89</td>
<td>0.79</td>
<td>1.43</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>0.46</td>
<td>0.71</td>
<td>0.79</td>
<td>1.38</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>23.3</td>
<td>4.04</td>
<td>0.85</td>
<td>1.69</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>20.6</td>
<td>2.78</td>
<td>0.81</td>
<td>1.39</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>6.4</td>
<td>4.47</td>
<td>1.30</td>
<td>1.95</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>6.4</td>
<td>4.01</td>
<td>0.85</td>
<td>1.50</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>15.0</td>
<td>2.91</td>
<td>0.85</td>
<td>1.39</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.000</td>
<td>0.011</td>
<td>0.187</td>
<td>0.802</td>
<td></td>
</tr>
</tbody>
</table>

Table A.11: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.1,.3,.5,1)$ and $\theta = 3$ (power$\approx84.9\%$)

† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta = \frac{1}{1}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>3.06</td>
<td>0.43</td>
<td>-0.84</td>
<td>0.379</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>2.78</td>
<td>0.07</td>
<td>-1.15</td>
<td>0.042</td>
</tr>
<tr>
<td>bias $\hat{\theta}$</td>
<td>NA</td>
<td>-1.02</td>
<td>-0.63</td>
<td>0.07</td>
<td>-0.478</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>NA</td>
<td>-1.02</td>
<td>-0.89</td>
<td>-0.84</td>
<td>-0.889</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>2.12</td>
<td>0.17</td>
<td>-0.84</td>
<td>0.114</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>0.58</td>
<td>0.69</td>
<td>0.62</td>
<td>1.92</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>0.68</td>
<td>0.69</td>
<td>0.53</td>
<td>1.92</td>
</tr>
<tr>
<td>Var $\hat{\theta}$</td>
<td>NA</td>
<td>8.98</td>
<td>6.33</td>
<td>1.54</td>
<td>5.41</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>8.98</td>
<td>4.75</td>
<td>0.62</td>
<td>4.07</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>0.82</td>
<td>1.14</td>
<td>0.62</td>
<td>1.69</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>9.9</td>
<td>0.88</td>
<td>1.33</td>
<td>2.06</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>8.4</td>
<td>0.70</td>
<td>1.84</td>
<td>1.92</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>10.0</td>
<td>6.73</td>
<td>1.54</td>
<td>5.64</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>10.0</td>
<td>5.53</td>
<td>1.33</td>
<td>4.86</td>
</tr>
<tr>
<td>$\hat{\theta}$'</td>
<td>NA</td>
<td>5.3</td>
<td>1.17</td>
<td>1.33</td>
<td>1.70</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.000</td>
<td>0.117</td>
<td>0.600</td>
<td>0.283</td>
<td></td>
</tr>
</tbody>
</table>

Table A.12: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.1,.3,.5,1)$ and $\theta = 5$ (power$\approx99.9\%$)

† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$n = \eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>5.99</td>
<td>3.64</td>
<td>2.75</td>
<td>-0.08</td>
<td>0.158</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>5.38</td>
<td>2.87</td>
<td>2.07</td>
<td>-0.22</td>
<td>-0.019</td>
</tr>
<tr>
<td>bias</td>
<td>1.58</td>
<td>1.01</td>
<td>0.95</td>
<td>0.04</td>
<td>0.104</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>1.58</td>
<td>1.00</td>
<td>0.83</td>
<td>-0.08</td>
<td>-0.011</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>1.58</td>
<td>3.35</td>
<td>2.49</td>
<td>-0.08</td>
<td>0.096</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>41.4</td>
<td>5.08</td>
<td>1.12</td>
<td>0.89</td>
<td>2.38</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>34.8</td>
<td>3.60</td>
<td>0.76</td>
<td>0.63</td>
<td>1.78</td>
</tr>
<tr>
<td>Var</td>
<td>12.9</td>
<td>3.70</td>
<td>2.98</td>
<td>1.19</td>
<td>1.51</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>12.9</td>
<td>3.59</td>
<td>2.11</td>
<td>0.89</td>
<td>1.21</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>12.9</td>
<td>4.62</td>
<td>1.11</td>
<td>0.89</td>
<td>1.62</td>
</tr>
<tr>
<td>MSE</td>
<td>77.3</td>
<td>18.3</td>
<td>8.67</td>
<td>0.90</td>
<td>2.40</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>63.7</td>
<td>11.8</td>
<td>5.02</td>
<td>0.68</td>
<td>1.78</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>15.4</td>
<td>4.7</td>
<td>3.88</td>
<td>1.20</td>
<td>1.52</td>
</tr>
<tr>
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<td>15.4</td>
<td>4.6</td>
<td>2.80</td>
<td>0.90</td>
<td>1.21</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>15.4</td>
<td>15.9</td>
<td>7.33</td>
<td>0.90</td>
<td>1.62</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.011</td>
<td>0.026</td>
<td>0.027</td>
<td>0.936</td>
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</tr>
</tbody>
</table>

Table A.13: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.1,.3,.5,1)$ and $\theta = 1$ (power$\approx$14.0$\%$)
† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$n = \eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>6.59</td>
<td>2.41</td>
<td>1.09</td>
<td>-0.46</td>
<td>0.670</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>5.79</td>
<td>1.55</td>
<td>0.35</td>
<td>-0.90</td>
<td>0.083</td>
</tr>
<tr>
<td>bias</td>
<td>1.00</td>
<td>-0.002</td>
<td>-0.11</td>
<td>0.08</td>
<td>0.068</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>1.00</td>
<td>-0.04</td>
<td>-0.43</td>
<td>-0.46</td>
<td>-0.314</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>1.00</td>
<td>2.14</td>
<td>0.89</td>
<td>-0.46</td>
<td>0.336</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>2.7</td>
<td>0.65</td>
<td>0.28</td>
<td>0.71</td>
<td>3.58</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>2.8</td>
<td>0.61</td>
<td>0.21</td>
<td>0.45</td>
<td>2.94</td>
</tr>
<tr>
<td>Var</td>
<td>10.3</td>
<td>4.95</td>
<td>4.86</td>
<td>1.38</td>
<td>3.14</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>10.3</td>
<td>4.57</td>
<td>2.79</td>
<td>0.71</td>
<td>2.34</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>10.3</td>
<td>0.88</td>
<td>0.47</td>
<td>0.71</td>
<td>2.15</td>
</tr>
<tr>
<td>MSE</td>
<td>46.2</td>
<td>6.46</td>
<td>1.47</td>
<td>0.92</td>
<td>4.03</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>36.3</td>
<td>3.62</td>
<td>0.34</td>
<td>1.26</td>
<td>2.95</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>11.3</td>
<td>4.95</td>
<td>4.87</td>
<td>1.38</td>
<td>3.14</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>11.3</td>
<td>4.57</td>
<td>2.98</td>
<td>0.92</td>
<td>2.43</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>11.3</td>
<td>5.48</td>
<td>1.26</td>
<td>0.92</td>
<td>2.26</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.044</td>
<td>0.179</td>
<td>0.196</td>
<td>0.581</td>
<td></td>
</tr>
</tbody>
</table>

Table A.14: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.1,.3,.5,1)$ and $\theta = 3$ (power$\approx$79.7$\%$)
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$\eta = $</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>5.00</td>
<td>0.93</td>
<td>-0.53</td>
<td>-1.17</td>
<td>0.884</td>
</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>4.23</td>
<td>0.06</td>
<td>-1.31</td>
<td>-1.85</td>
<td>0.075</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>-0.09</td>
<td>-0.64</td>
<td>-0.64</td>
<td>0.04</td>
<td>-0.481</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>-0.09</td>
<td>-0.78</td>
<td>-1.46</td>
<td>-1.17</td>
<td>-0.901</td>
</tr>
<tr>
<td>$\hat{\theta}'''$</td>
<td>-0.09</td>
<td>0.74</td>
<td>-0.65</td>
<td>-1.17</td>
<td>0.039</td>
</tr>
</tbody>
</table>

| $\delta$ | 1.9   | 1.12  | 0.50  | 0.60  | 4.45    |
| $\tilde{\delta}$ | 2.4   | 1.10  | 0.40  | 0.42  | 4.42    |
| $\delta'$ | 13.4  | 7.31  | 7.03  | 1.64  | 7.50    |
| $\delta''$ | 13.4  | 6.21  | 3.07  | 0.60  | 5.95    |
| $\delta'''$ | 13.4  | 1.45  | 0.60  | 0.60  | 3.39    |

| MSE $\hat{\theta}$ | 26.9  | 1.98  | 0.78  | 1.97  | 5.23    |
| MSE $\tilde{\theta}$ | 20.2  | 1.10  | 2.12  | 3.88  | 4.42    |
| MSE $\theta'$ | 13.4  | 7.72  | 7.43  | 1.64  | 7.73    |
| MSE $\theta''$ | 13.4  | 6.82  | 5.19  | 1.97  | 6.76    |
| MSE $\theta'''$ | 13.4  | 2.00  | 1.11  | 1.97  | 3.39    |

| Prob($\eta$) | 0.143 | 0.482 | 0.256 | 0.120 |

Table A.15: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.1,.3,.5,1)$ and $\theta = 5$ (power$\approx99.7\%$)
† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta = $</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>6.37</td>
<td>3.91</td>
<td>2.75</td>
<td>-0.08</td>
<td>0.133</td>
</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>5.76</td>
<td>3.17</td>
<td>2.12</td>
<td>-0.20</td>
<td>-0.022</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>1.68</td>
<td>1.08</td>
<td>0.91</td>
<td>0.04</td>
<td>0.095</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>1.68</td>
<td>1.08</td>
<td>0.82</td>
<td>-0.08</td>
<td>-0.014</td>
</tr>
<tr>
<td>$\hat{\theta}'''$</td>
<td>1.68</td>
<td>3.59</td>
<td>2.48</td>
<td>-0.08</td>
<td>0.086</td>
</tr>
</tbody>
</table>

| $\theta$ | 44.7  | 4.24  | 1.48  | 0.90  | 2.11    |
| $\tilde{\theta}$ | 37.9  | 3.11  | 1.03  | 0.66  | 1.60    |
| $\theta'$ | 13.9  | 3.44  | 3.06  | 1.19  | 1.44    |
| $\theta''$ | 13.9  | 3.38  | 2.29  | 0.90  | 1.15    |
| $\theta'''$ | 13.9  | 3.84  | 1.44  | 0.90  | 1.53    |

| $\delta$ | 85.3  | 19.5  | 9.05  | 0.91  | 2.12    |
| $\tilde{\delta}$ | 71.1  | 13.1  | 5.53  | 0.70  | 1.60    |
| $\delta'$ | 16.8  | 4.6   | 3.90  | 1.19  | 1.45    |
| $\delta''$ | 16.8  | 4.6   | 2.95  | 0.91  | 1.15    |
| $\delta'''$ | 16.8  | 16.7  | 7.57  | 0.91  | 1.54    |

| Prob($\eta$) | 0.007 | 0.021 | 0.028 | 0.944 |

Table A.16: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=(.1,.3,.5,1)$ and $\theta = 1$ (power$\approx14.9\%$)
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$\eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>7.01</td>
<td>2.56</td>
<td>1.14</td>
<td>-0.44</td>
<td>0.603</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>6.21</td>
<td>1.75</td>
<td>0.45</td>
<td>-0.84</td>
<td>0.065</td>
</tr>
<tr>
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Table A.17: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=\{.1,.3,.5,1\}$ and $\theta = 3$ (power≈81.6%)
† based on 100,000 simulations

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Table A.18: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=\{.1,.3,.5,1\}$ and $\theta = 5$ (power≈99.8%)
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$\eta$</th>
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<th>overall</th>
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<tbody>
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<td>0.45</td>
<td>0.32</td>
<td>0.61</td>
<td>1.09</td>
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<td>2.42</td>
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<td>0.99</td>
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Table A.19: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,.1)$ and $\theta = 1$ (power$\approx$16.5%)  
† based on 100,000 simulations

<table>
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<td>0.13</td>
<td>0.26</td>
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<td>1.18</td>
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<td>1.89</td>
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<tr>
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<td>2.53</td>
<td>2.04</td>
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<td>2.49</td>
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<tr>
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<td>0.64</td>
<td>0.71</td>
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<td>1.55</td>
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Table A.20: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,.1)$ and $\theta = 3$ (power$\approx$83.9%)  
† based on 100,000 simulations

42
<table>
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<td>1.54</td>
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<tr>
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<td>1.89</td>
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<td>1.71</td>
</tr>
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<td>6.05</td>
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<td>5.62</td>
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<td>4.37</td>
<td>5.64</td>
<td>7.65</td>
<td>5.17</td>
</tr>
<tr>
<td>$\tilde{\theta}''$</td>
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<td>1.84</td>
<td>4.27</td>
<td>7.65</td>
<td>4.51</td>
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Table A.21: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,1)$ and $\theta = 5$ (power≈99.8%) † based on 100,000 simulations

<table>
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<tr>
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<th>4</th>
<th>overall</th>
</tr>
</thead>
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<td>0.82</td>
<td>0.22</td>
<td>0.280</td>
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</tr>
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<td>0.45</td>
<td>0.19</td>
<td>0.58</td>
<td>1.23</td>
</tr>
<tr>
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<td>3.06</td>
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<td>1.35</td>
<td>1.01</td>
<td>0.73</td>
<td>0.90</td>
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<tr>
<td>$\tilde{\theta}''$</td>
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<td>0.59</td>
<td>0.44</td>
<td>0.73</td>
<td>0.98</td>
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<tr>
<td>$\bar{\theta}$</td>
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<td>0.77</td>
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<tr>
<td>$\tilde{\theta}$</td>
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<td>1.23</td>
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<tr>
<td>MSE $\tilde{\theta}$</td>
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<td>3.91</td>
<td>3.53</td>
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</tr>
<tr>
<td>$\tilde{\theta}'$</td>
<td>2.04</td>
<td>1.75</td>
<td>1.24</td>
<td>0.77</td>
<td>0.91</td>
</tr>
<tr>
<td>$\tilde{\theta}''$</td>
<td>2.04</td>
<td>3.82</td>
<td>2.06</td>
<td>0.77</td>
<td>0.98</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.076</td>
<td>0.027</td>
<td>0.027</td>
<td>0.070</td>
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Table A.22: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,1)$ and $\theta = 1$ (power≈14.3%) † based on 100,000 simulations
### Table A.23: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T= (.5, .7, .9, 1)$ and $\theta = 3$ (power=78.5%)

<table>
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<th>overall</th>
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<tbody>
<tr>
<td></td>
<td>$\hat{\theta}$</td>
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<td>-1.14</td>
<td>0.294</td>
</tr>
<tr>
<td>bias</td>
<td>$\hat{\delta}$</td>
<td>0.96</td>
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<td>-0.39</td>
<td>-1.34</td>
<td>0.076</td>
</tr>
<tr>
<td></td>
<td>$\hat{\theta}'$</td>
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<td>-0.17</td>
<td>0.46</td>
<td>-0.071</td>
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<tr>
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<td>-0.82</td>
<td>-1.00</td>
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<td>Var</td>
<td>$\hat{\delta}$</td>
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<td>0.10</td>
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<tr>
<td></td>
<td>$\hat{\delta}$</td>
<td>0.87</td>
<td>0.16</td>
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<tr>
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<td>1.47</td>
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<td>$\hat{\delta}$</td>
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<td>2.08</td>
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<tr>
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<td>0.135</td>
<td>0.116</td>
<td>0.259</td>
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</tbody>
</table>

† based on 100,000 simulations

### Table A.24: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T= (.5, .7, .9, 1)$ and $\theta = 5$ (power=99.7%)

<table>
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<tbody>
<tr>
<td></td>
<td>$\hat{\theta}$</td>
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<td>-2.62</td>
<td>0.087</td>
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<tr>
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<td>-2.15</td>
<td>-2.87</td>
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</tr>
<tr>
<td></td>
<td>$\hat{\theta}'$</td>
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<td>-0.61</td>
<td>-0.57</td>
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<td>-0.410</td>
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<tr>
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<td>-2.09</td>
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<td>-2.62</td>
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<tr>
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<td>1.96</td>
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<tr>
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<tr>
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<td>7.08</td>
<td>1.66</td>
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<tr>
<td>MSE</td>
<td>$\hat{\delta}$</td>
<td>1.72</td>
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<td>4.79</td>
<td>8.43</td>
<td>1.91</td>
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<tr>
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<td>7.18</td>
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<tr>
<td></td>
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<td>5.95</td>
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<td>7.08</td>
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<td>0.087</td>
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<td>0.006</td>
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</tbody>
</table>

† based on 100,000 simulations
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<tr>
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<th>3</th>
<th>4</th>
<th>overall</th>
</tr>
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<td>0.24</td>
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<td>0.62</td>
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<td>-0.103</td>
</tr>
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<td>$\hat{\theta}''$</td>
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<td>1.74</td>
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<td>0.66</td>
<td>0.32</td>
<td>0.72</td>
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<td>0.26</td>
<td>0.58</td>
<td>1.21</td>
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<tr>
<td>Var $\hat{\theta}$</td>
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<td>2.97</td>
<td>2.89</td>
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</tr>
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<td>1.41</td>
<td>1.04</td>
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<td>0.89</td>
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<td>0.70</td>
<td>0.52</td>
<td>0.72</td>
<td>0.98</td>
</tr>
<tr>
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<td>8.33</td>
<td>4.74</td>
<td>3.03</td>
<td>0.76</td>
<td>1.45</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>7.08</td>
<td>3.60</td>
<td>2.16</td>
<td>0.66</td>
<td>1.21</td>
</tr>
<tr>
<td>MSE $\hat{\theta}$</td>
<td>2.11</td>
<td>3.75</td>
<td>3.63</td>
<td>2.26</td>
<td>2.35</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
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<td>1.29</td>
<td>0.76</td>
<td>0.90</td>
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<td>2.11</td>
<td>3.74</td>
<td>2.01</td>
<td>0.76</td>
<td>0.99</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.064</td>
<td>0.031</td>
<td>0.035</td>
<td>0.870</td>
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Table A.25: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,1)$ and $\theta = 1$ (power=14.8%) 
† based on 100,000 simulations

<table>
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<th>4</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
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<td>-1.18</td>
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<td>0.002</td>
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<td>-1.37</td>
<td>0.076</td>
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<tr>
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<td>-0.21</td>
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<td>-0.077</td>
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<tr>
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<td>-0.78</td>
<td>-1.01</td>
<td>-1.18</td>
<td>-0.699</td>
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<td>-0.47</td>
<td>-1.18</td>
<td>-0.487</td>
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<td>0.16</td>
<td>0.11</td>
<td>0.34</td>
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<tr>
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<td>0.17</td>
<td>0.10</td>
<td>0.27</td>
<td>1.46</td>
</tr>
<tr>
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<td>5.01</td>
<td>4.71</td>
<td>3.46</td>
<td>3.78</td>
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<td>3.01</td>
<td>1.70</td>
<td>1.20</td>
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<td>0.34</td>
<td>0.41</td>
<td>0.34</td>
<td>1.75</td>
</tr>
<tr>
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<td>2.20</td>
<td>0.23</td>
<td>0.14</td>
<td>1.74</td>
<td>1.49</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.91</td>
<td>0.17</td>
<td>0.30</td>
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<td>1.47</td>
</tr>
<tr>
<td>MSE $\hat{\theta}$</td>
<td>3.11</td>
<td>5.03</td>
<td>4.75</td>
<td>3.68</td>
<td>3.79</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>3.11</td>
<td>2.31</td>
<td>2.22</td>
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<td>2.52</td>
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<td>0.34</td>
<td>0.63</td>
<td>1.74</td>
<td>1.99</td>
</tr>
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<td>Prob($\eta$)</td>
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<td>0.138</td>
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Table A.26: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,1)$ and $\theta = 3$ (power=80.3%) 
† based on 100,000 simulations
<table>
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<th>4</th>
<th>overall</th>
</tr>
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<td>$\hat{\theta}$</td>
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<td>1.63</td>
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<td>0.20</td>
<td>0.13</td>
<td>1.88</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
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<td>7.11</td>
<td>6.68</td>
<td>4.15</td>
<td>4.13</td>
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<tr>
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<td>1.50</td>
<td>2.33</td>
<td>3.78</td>
<td>7.24</td>
<td>1.64</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.67</td>
<td>3.18</td>
<td>4.92</td>
<td>8.57</td>
<td>1.88</td>
</tr>
<tr>
<td>MSE</td>
<td>4.02</td>
<td>7.47</td>
<td>6.92</td>
<td>4.30</td>
<td>4.32</td>
</tr>
<tr>
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<td>5.78</td>
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<td>4.21</td>
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<td>2.80</td>
<td>4.54</td>
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<td>3.96</td>
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<td>0.070</td>
<td>0.022</td>
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Table A.27: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=(.5,.7,.9,1)$ and $\theta = 5$ (power=99.8%)
† based on 100,000 simulations

<table>
<thead>
<tr>
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<th>3</th>
<th>overall</th>
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<td>0.08</td>
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<td>0.54</td>
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<td>-0.051</td>
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<tr>
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<td>1.78</td>
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<td>0.61</td>
<td>0.87</td>
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</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>2.14</td>
<td>0.54</td>
<td>0.78</td>
<td>1.04</td>
</tr>
<tr>
<td>Var</td>
<td>4.51</td>
<td>1.55</td>
<td>1.33</td>
<td>1.35</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>4.51</td>
<td>1.51</td>
<td>0.87</td>
<td>0.92</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>4.51</td>
<td>0.74</td>
<td>0.87</td>
<td>1.02</td>
</tr>
<tr>
<td>MSE</td>
<td>35.7</td>
<td>6.88</td>
<td>0.88</td>
<td>1.18</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>31.9</td>
<td>5.75</td>
<td>0.80</td>
<td>1.04</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>6.0</td>
<td>1.84</td>
<td>1.34</td>
<td>1.36</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>6.0</td>
<td>1.80</td>
<td>0.88</td>
<td>0.93</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>6.0</td>
<td>3.91</td>
<td>0.88</td>
<td>1.02</td>
</tr>
<tr>
<td>Prob(\eta)</td>
<td>0.0009</td>
<td>0.044</td>
<td>0.955</td>
<td></td>
</tr>
</tbody>
</table>

Table A.28: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric OBF type boundary ($\alpha = .05$) is used with $T=(.33,.67,.1)$ and $\theta = 1$ (power=16.6%)
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>η =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ</td>
<td>4.08</td>
<td>0.97</td>
<td>-0.62</td>
<td>0.209</td>
</tr>
<tr>
<td>θ̂</td>
<td>3.76</td>
<td>0.73</td>
<td>-0.77</td>
<td>0.016</td>
</tr>
<tr>
<td>bias</td>
<td>-0.12</td>
<td>-0.34</td>
<td>0.15</td>
<td>-0.777</td>
</tr>
<tr>
<td>θ'</td>
<td>-0.12</td>
<td>-0.39</td>
<td>-0.62</td>
<td>-0.507</td>
</tr>
<tr>
<td>θ''</td>
<td>-0.12</td>
<td>0.46</td>
<td>-0.62</td>
<td>-0.121</td>
</tr>
<tr>
<td>Var</td>
<td>0.36</td>
<td>0.45</td>
<td>0.58</td>
<td>1.50</td>
</tr>
<tr>
<td>θ</td>
<td>0.41</td>
<td>0.42</td>
<td>0.50</td>
<td>1.34</td>
</tr>
<tr>
<td>θ̂</td>
<td>5.72</td>
<td>3.10</td>
<td>1.73</td>
<td>2.50</td>
</tr>
<tr>
<td>θ̂'</td>
<td>5.72</td>
<td>2.77</td>
<td>0.58</td>
<td>1.71</td>
</tr>
<tr>
<td>θ̂''</td>
<td>5.72</td>
<td>1.02</td>
<td>0.58</td>
<td>1.19</td>
</tr>
<tr>
<td>MSE</td>
<td>5.72</td>
<td>2.92</td>
<td>0.97</td>
<td>1.97</td>
</tr>
<tr>
<td>θ</td>
<td>17.0</td>
<td>1.40</td>
<td>0.97</td>
<td>1.54</td>
</tr>
<tr>
<td>θ̂</td>
<td>14.5</td>
<td>0.96</td>
<td>1.09</td>
<td>1.34</td>
</tr>
<tr>
<td>θ̂'</td>
<td>5.7</td>
<td>3.21</td>
<td>1.75</td>
<td>2.51</td>
</tr>
<tr>
<td>θ̂''</td>
<td>5.7</td>
<td>2.92</td>
<td>0.97</td>
<td>1.97</td>
</tr>
<tr>
<td>Prob(η)</td>
<td>0.023</td>
<td>0.453</td>
<td>0.524</td>
<td></td>
</tr>
</tbody>
</table>

Table A.29: Table of the bias, variance and MSE of different estimators of θ when a symmetric OBF type boundary (α = .05) is used with T=(.33,.67,.1) and θ = 3 (power≈84.5%)  
† based on 100,000 simulations

<table>
<thead>
<tr>
<th>η =</th>
<th>1</th>
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<th>3</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ</td>
<td>2.40</td>
<td>-0.14</td>
<td>-1.64</td>
<td>0.296</td>
</tr>
<tr>
<td>θ̂</td>
<td>2.10</td>
<td>-0.42</td>
<td>-1.85</td>
<td>0.019</td>
</tr>
<tr>
<td>bias</td>
<td>-0.97</td>
<td>-0.32</td>
<td>0.10</td>
<td>-0.426</td>
</tr>
<tr>
<td>θ'</td>
<td>-0.97</td>
<td>-0.63</td>
<td>-1.64</td>
<td>-0.753</td>
</tr>
<tr>
<td>θ''</td>
<td>-0.97</td>
<td>-0.33</td>
<td>-1.64</td>
<td>-0.537</td>
</tr>
<tr>
<td>Var</td>
<td>0.65</td>
<td>0.37</td>
<td>0.43</td>
<td>2.67</td>
</tr>
<tr>
<td>θ</td>
<td>0.78</td>
<td>0.80</td>
<td>0.38</td>
<td>2.01</td>
</tr>
<tr>
<td>θ̂</td>
<td>8.03</td>
<td>3.77</td>
<td>2.18</td>
<td>4.64</td>
</tr>
<tr>
<td>θ̂'</td>
<td>8.03</td>
<td>2.65</td>
<td>0.43</td>
<td>3.69</td>
</tr>
<tr>
<td>θ̂''</td>
<td>8.03</td>
<td>1.39</td>
<td>0.43</td>
<td>2.84</td>
</tr>
<tr>
<td>MSE</td>
<td>8.96</td>
<td>3.87</td>
<td>2.19</td>
<td>4.82</td>
</tr>
<tr>
<td>θ</td>
<td>8.96</td>
<td>3.04</td>
<td>3.11</td>
<td>4.26</td>
</tr>
<tr>
<td>θ̂</td>
<td>8.96</td>
<td>1.90</td>
<td>3.11</td>
<td>3.12</td>
</tr>
<tr>
<td>Prob(η)</td>
<td>0.206</td>
<td>0.737</td>
<td>0.057</td>
<td></td>
</tr>
</tbody>
</table>

Table A.30: Table of the bias, variance and MSE of different estimators of θ when a symmetric OBF type boundary (α = .05) is used with T=(.33,.67,.1) and θ = 5 (power≈99.9%)  
† based on 100,000 simulations
<table>
<thead>
<tr>
<th>$\eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>3.25</td>
<td>2.17</td>
<td>-0.13</td>
<td>0.135</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>2.88</td>
<td>1.75</td>
<td>-0.24</td>
<td>-0.001</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.82</td>
<td>0.67</td>
<td>0.09</td>
<td>0.149</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.82</td>
<td>0.60</td>
<td>-0.13</td>
<td>-0.052</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.82</td>
<td>1.77</td>
<td>-0.13</td>
<td>0.003</td>
</tr>
<tr>
<td>$\psi$</td>
<td>3.88</td>
<td>0.70</td>
<td>0.82</td>
<td>1.67</td>
</tr>
<tr>
<td>$\hat{\psi}$</td>
<td>3.39</td>
<td>0.55</td>
<td>0.62</td>
<td>1.33</td>
</tr>
<tr>
<td>$\hat{\psi}$</td>
<td>2.70</td>
<td>1.98</td>
<td>1.40</td>
<td>1.53</td>
</tr>
<tr>
<td>$\hat{\psi}$</td>
<td>2.70</td>
<td>1.53</td>
<td>0.82</td>
<td>1.00</td>
</tr>
<tr>
<td>$\hat{\psi}$</td>
<td>2.70</td>
<td>0.80</td>
<td>0.82</td>
<td>1.10</td>
</tr>
<tr>
<td>MSE</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>14.4</td>
<td>5.40</td>
<td>0.83</td>
<td>1.68</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>11.7</td>
<td>3.63</td>
<td>0.68</td>
<td>1.33</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.4</td>
<td>2.44</td>
<td>1.41</td>
<td>1.55</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.4</td>
<td>1.89</td>
<td>0.83</td>
<td>1.00</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.4</td>
<td>3.01</td>
<td>0.83</td>
<td>1.10</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.047</td>
<td>0.047</td>
<td>0.906</td>
<td></td>
</tr>
</tbody>
</table>

Table A.31: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.33,.67,.1)$ and $\theta = 1$ (power≈14.1%) † based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>2.03</td>
<td>0.55</td>
<td>-0.76</td>
<td>0.453</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.68</td>
<td>0.12</td>
<td>-1.06</td>
<td>0.100</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>-0.15</td>
<td>-0.29</td>
<td>0.15</td>
<td>-0.071</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>-0.15</td>
<td>-0.56</td>
<td>-0.76</td>
<td>-0.521</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>-0.15</td>
<td>0.28</td>
<td>-0.76</td>
<td>-0.270</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.79</td>
<td>0.32</td>
<td>0.55</td>
<td>1.89</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.01</td>
<td>0.32</td>
<td>0.41</td>
<td>1.84</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.89</td>
<td>3.53</td>
<td>1.78</td>
<td>2.96</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.89</td>
<td>2.19</td>
<td>0.55</td>
<td>2.09</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.89</td>
<td>0.64</td>
<td>0.55</td>
<td>1.75</td>
</tr>
<tr>
<td>MSE</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>4.92</td>
<td>0.62</td>
<td>1.13</td>
<td>2.09</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.84</td>
<td>0.33</td>
<td>1.53</td>
<td>1.85</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.92</td>
<td>3.61</td>
<td>1.80</td>
<td>2.97</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.92</td>
<td>2.51</td>
<td>1.13</td>
<td>2.36</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>3.92</td>
<td>0.72</td>
<td>1.13</td>
<td>1.83</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.295</td>
<td>0.298</td>
<td>0.407</td>
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</tr>
</tbody>
</table>

Table A.32: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Pocock type boundary ($\alpha = .05$) is used with $T=(.33,.67,.1)$ and $\theta = 3$ (power≈79.2%) † based on 100,000 simulations
| $\eta = 1$ | $\theta$ | $-0.83$ | $1.78$ | 0.323 |
| $\theta$ | 0.54 | 1.25 | 2.19 | 0.026 |
| bias $\theta$ | -0.61 | -0.46 | 0.14 | -0.546 |
| $\theta'$ | -0.61 | -1.29 | -1.78 | -0.807 |
| $\theta''$ | -0.61 | -0.94 | -1.78 | -0.723 |
| $\theta$ | 1.54 | 0.61 | 0.41 | 1.91 |
| $\theta$ | 1.92 | 0.69 | 0.35 | 2.32 |
| Var $\theta$ | 5.62 | 4.67 | 2.18 | 5.30 |
| $\theta'$ | 5.62 | 2.08 | 0.41 | 4.73 |
| $\theta''$ | 5.62 | 0.86 | 0.41 | 4.38 |
| $\theta$ | 2.17 | 1.31 | 3.59 | 2.01 |
| $\theta$ | 2.21 | 2.25 | 5.16 | 2.32 |
| MSE $\theta$ | 5.99 | 4.88 | 2.20 | 5.60 |
| $\theta'$ | 5.99 | 3.76 | 3.59 | 5.38 |
| $\theta''$ | 5.99 | 1.75 | 3.59 | 4.90 |
| Prob($\eta$) | 0.730 | 0.237 | 0.033 |

Table A.33: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric Poisson type boundary ($\alpha = .05$) is used with $T=(.33, .67, 1)$ and $\theta = 5$ (power≈99.7%)  
† based on 100,000 simulations.

| $\eta = 1$ | $\theta$ | 3.44 | 2.17 | -0.13 | 0.119 |
| $\theta$ | 3.08 | 1.77 | -0.23 | -0.006 |
| bias $\theta$ | 0.85 | 0.63 | 0.09 | 0.145 |
| $\theta'$ | 0.85 | 0.57 | -0.13 | -0.055 |
| $\theta''$ | 0.85 | 1.74 | -0.13 | 0.005 |
| $\theta$ | 3.85 | 0.71 | 0.83 | 1.59 |
| $\theta$ | 3.39 | 0.56 | 0.64 | 1.28 |
| Var $\theta$ | 2.79 | 1.88 | 1.41 | 1.52 |
| $\theta'$ | 2.79 | 1.51 | 0.83 | 0.99 |
| $\theta''$ | 2.79 | 0.80 | 0.83 | 1.09 |
| $\theta$ | 15.7 | 5.41 | 0.84 | 1.61 |
| $\theta$ | 12.9 | 3.70 | 0.69 | 1.28 |
| MSE $\theta$ | 3.5 | 2.27 | 1.42 | 1.54 |
| $\theta'$ | 3.5 | 1.84 | 0.84 | 0.99 |
| $\theta''$ | 3.5 | 3.84 | 0.84 | 1.09 |
| Prob($\eta$) | 0.036 | 0.051 | 0.913 |

Table A.34: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T=(.33, .67, 1)$ and $\theta = 1$ (power≈14.9%)  
† based on 100,000 simulations.
Table A.35: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T= (.33, .67, 1)$ and $\theta = 3$ (power=81.1%)
† based on 100,000 simulations

<table>
<thead>
<tr>
<th>$\eta$ =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\theta}$</td>
<td>2.18</td>
<td>0.58</td>
<td>-0.75</td>
<td>0.434</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>1.84</td>
<td>0.16</td>
<td>-1.03</td>
<td>0.091</td>
</tr>
<tr>
<td>bias $\tilde{\theta}$</td>
<td>-0.15</td>
<td>-0.30</td>
<td>0.15</td>
<td>-0.073</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>-0.15</td>
<td>-0.54</td>
<td>-0.75</td>
<td>-0.529</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>-0.15</td>
<td>0.29</td>
<td>-0.75</td>
<td>-0.255</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.73</td>
<td>0.34</td>
<td>0.55</td>
<td>1.90</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.93</td>
<td>0.33</td>
<td>0.41</td>
<td>1.82</td>
</tr>
<tr>
<td>$\text{Var} \hat{\theta}$</td>
<td>4.01</td>
<td>3.47</td>
<td>1.78</td>
<td>2.94</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>4.01</td>
<td>2.23</td>
<td>0.55</td>
<td>2.04</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>4.01</td>
<td>0.68</td>
<td>0.55</td>
<td>1.68</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>5.49</td>
<td>0.67</td>
<td>1.12</td>
<td>2.09</td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>4.30</td>
<td>0.35</td>
<td>1.48</td>
<td>1.83</td>
</tr>
<tr>
<td>$\text{MSE} \hat{\theta}$</td>
<td>4.03</td>
<td>3.55</td>
<td>1.80</td>
<td>2.95</td>
</tr>
<tr>
<td>$\hat{\theta}'$</td>
<td>4.03</td>
<td>2.52</td>
<td>1.12</td>
<td>2.32</td>
</tr>
<tr>
<td>$\hat{\theta}''$</td>
<td>4.03</td>
<td>0.76</td>
<td>1.12</td>
<td>1.74</td>
</tr>
<tr>
<td>Prob($\eta$)</td>
<td>0.256</td>
<td>0.329</td>
<td>0.416</td>
<td></td>
</tr>
</tbody>
</table>

Table A.36: Table of the bias, variance and MSE of different estimators of $\theta$ when a symmetric uniform type boundary ($\alpha = .05$) is used with $T= (.33, .67, 1)$ and $\theta = 5$ (power=99.8%)
† based on 100,000 simulations