ON FIRST ORDER ASYMPTOTIC EXPANSIONS FOR MARTINGALES

Siu-Kai Chan
Department of Biostatistics
University of Wisconsin - Madison

Per Mykland
Department of Statistics
University of Chicago

MADISON, WISCONSIN
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by

Siu-Kai Chan
Department of Biostatistics
University of Wisconsin
Madison, WI 53792

Per Mykland
Department of Statistics
The University of Chicago
Chicago, IL 60637

Abstract

Conditions under which a one step triangular array asymptotic expansion for martingales exists are developed. The conditions are different from and easier to verify than earlier ones.

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1 Research supported in part by National Eye Institute grant NEI-T32-EY07119-04 and National Science Foundation Grant No. DMS 93-05601.

AMS 1991 subject classifications. Primary 60F99, 60G42, 60G44, 62E20;

Key words and phrases. Edgeworth-expansions, martingales, Skorokhod embedding, time series.

Running head. Expansions for martingales.
1. Introduction

The validity of the Edgeworth expansion for martingales has been studied by Mykland (1990, 1992, 1993a, 1993b). In those papers, conditions for the asymptotic expansion of continuous and discrete time martingales arrays are developed. Applications for these results can be found in inference problems involving parametric models, time series and survival analysis. When it comes to verifying the conditions in practice, however, one would typically use the procedure outlined in connection with Proposition 2 in Mykland (1993a). This has the drawback that one is either restricted to non-triangular arrays, or one has to impose additional asymptotic negligibility conditions.

In this paper, we further develop the results in Mykland (1993a) and show that more easily verifiable conditions can be obtained without the above drawback. The alternative conditions we obtain here for asymptotic expansion of martingales are analogous to Proposition 2 of Mykland (1993a) and can be applied to discrete and continuous time triangular martingale arrays. The topology we use for the convergence of the expansion is the same as Mykland (1992, 1993a) which has a smoothing effect for discrete distributions.

2. Variation Processes

Let $\ell^n_t$ be a triangular array of zero mean martingales with respect to a filtration $\mathcal{F}^n_t$, where $n = 1, 2, \ldots$, and $t$ is either discrete with $t = 0, 1, 2, \ldots, T_n$ or continuous with $t \in [0, T_n]$. Let $\ell^n_t$ be càdlàg when $t$ is continuous. (Càdlàg means continuous from the right with limits from the left.)

The $k$th order optional variation process of $\ell^n_t$ is defined as

$$\left[\ell^n_0, \ldots, \ell^n_t\right]_t = \sum_{i=1}^{t} (\ell^n_i - \ell^n_{i-1})^k$$

(2.1)

when $t$ is discrete, and as

$$\left[\ell^n_0, \ldots, \ell^n_t\right]_t = \lim_{\max(t_i - t_{i-1}) \downarrow 0} \sum_{i=1}^{\max(t_i - t_{i-1})} (\ell^n_i - \ell^n_{t_{i-1}})^k$$

(2.1a)

when $t$ is continuous, where \(\{0 = t_0, t_1, t_2, \ldots\}\) is a partition of \([0, t]\).

Also the $k$th order predictable variation process of $\ell^n_t$, denoted by

$$\left[\ell^n_0, \ldots, \ell^n_t\right]_t,$$

is defined as the compensator of the $k$th order optional variation process. For the discrete $t$ case, the predictable variation can be written as

$$\left[\ell^n_0, \ldots, \ell^n_t\right]_t = \sum_{i=1}^{t} E((\ell^n_i - \ell^n_{i-1})^k \mid \mathcal{F}^n_{i-1})$$

(2.2)
Similarly, if $m^n_t$ is a second martingale array adapted to $\mathcal{F}^n_t$ with the same properties as $\ell^n_t$, variation processes associated with $\ell^n_t$ and $m^n_t$ are defined as

$$\left[\ell^n, \ldots, \ell^n, m^n\right]_t = \sum_{i=1}^t (\ell^n_{t_i} - \ell^n_{t_{i-1}})^{k-1}(m^n_{t_i} - m^n_{t_{i-1}})$$ (2.3)

when $t$ is discrete and as

$$\left[\ell^n, \ldots, \ell^n, m^n\right]_t = \lim_{\max(t_{i-1}, t_{i-1}) \to t} \sum_{i=1}^{\max(t_i, t_{i-1})} (\ell^n_{t_i} - \ell^n_{t_{i-1}})^{k-1}(m^n_{t_i} - m^n_{t_{i-1}})$$ (2.3a)

when $t$ is continuous.

The compensator of (2.3) or (2.3a) is denoted by replacing the square brackets by angle brackets. When $t$ is discrete this can be written as

$$\langle \ell^n, \ldots, \ell^n, m^n \rangle_t = \sum_{i=1}^t E((\ell^n_{t_i} - \ell^n_{t_{i-1}})^{k-1}(m^n_{t_i} - m^n_{t_{i-1}}) | \mathcal{F}^n_{t_i})$$ (2.4)

3. Conditions for the Validity of Expansions

Let $c_n$ be some nonrandom sequence such that $\ell^n_{T_n} = O_p(c_n^{1/2})$, and $r_n$ be nonrandom and $o(1)$. Typically, $r_n$ is equal to $n^{-1/2}$, and $c_n$ is equal to $n$ (though this is not necessarily true in some applications). Also let $\sigma_n^2$ be some random sequence which will be used to studentize $\ell^n_{T_n}$. The present results are for obtaining an expansion for the distribution function

$$F_n(x) = P\left(\frac{\ell^n_{T_n}}{\sigma_n} \leq x \mid \sigma_n^2 > 0\right).$$ (3.1)

The conditions required for the validity of the expansion are

1. Integrability condition for the 4th order variation:

$$E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n} = O(c_n^2 r_n^2).$$ (3.2)

2. Integrability conditions for the square variation:

For $(\ell^n, \ell^n)_{T_n}$ being either $[\ell^n, \ell^n]_{T_n}$ or $(\ell^n, \ell^n)_{T_n}$, there are constants $b^2$, $\bar{k}$ and $\underline{k}$ such that $\infty \geq \bar{k} > b^2 > \underline{k} > 0$ and

$$r_n^{-1}\left((\ell^n, \ell^n)_{T_n}/c_n - b^2 I_{\{\frac{(\ell^n, \ell^n)_{T_n}}{c_n} \geq \bar{k}\}}\right) \text{ is uniformly integrable,}$$ (3.3)

where $I$ is the indicator function and

$$P\left(\frac{\bar{k}}{c_n} \geq \frac{\ell^n_{T_n}}{c_n} \geq \frac{\underline{k}}{c_n}\right) = 1 - o(r_n)$$ (3.4)
3. Integrability conditions for $\hat{\sigma}_n^2$

There are measurable sets $D_n^*$ and constants $b_n^2$ and $\delta > 0$ so that

$$\sup_n E \left[ r_n^{-1} \left| \frac{\hat{\sigma}_n^2}{c_n} - b_n^2 \right| \right]^{1+\delta} I_{D_n^*} < \infty$$  \hspace{1cm} (3.5)

and so that

$$P(D_n^*) = 1 - o(r_n)$$  \hspace{1cm} (3.6)

4. Asymptotic condition for the third order variation:

For some nonrandom $\mu_3$,

$$r_n^{-1} c_n^{-3/2} [e_n, e_n, e_n]_{T_n} - \mu_3 = o_p(1)$$  \hspace{1cm} (3.7)

5. The central limit condition:

For $(\ell^n, \ell^n)_{T_n}$ being either $[\ell^n, \ell^n]_{T_n}$ or $(\ell^n, \ell^n)_{T_n}$, whenever

$$\left( b_n^{-1} \frac{\ell^n_{T_n}}{\sqrt{c_n}}, r_n^{-1} \left( \frac{(\ell^n, \ell^n)_{T_n}}{c_n} - b^2 \right), r_n^{-1} \left( \frac{\hat{\sigma}_n^2}{c_n} - b_n^2 \right) \right)$$  \hspace{1cm} (3.8)

converges jointly in distribution to some $(Z, \xi, \xi_*)$ as $n$ goes to infinity through a subset of the integers, then $E(\xi | Z)$ and $E(\xi_* | Z)$ are uniquely defined up to a set of measure zero and are independent of the subsequence through which the convergence takes place.

$\xi_o, \psi_o$ will be used respectively to denote $\xi$ and a version of $b^{-2} E(\xi | Z)$ when $(\ell^n, \ell^n)_{T_n}$ is $[\ell^n, \ell^n]_{T_n}$ and $\xi_p, \psi_p$ will be used when $(\ell^n, \ell^n)_{T_n}$ is $(\ell^n, \ell^n)_{T_n}$. Also, $\psi_*$ is used to denote a version of $b_*^{-2} E(\xi_* | Z)$.

4. The Results

The main theorem is as follows:

**Theorem 1.** Suppose that conditions (3.2) – (3.4) and (3.7) above are satisfied by the zero mean martingale array $\ell_n^t$, where $n = 1, 2, \ldots$, and $t$ is either discrete with $t = 0, 1, 2, \ldots, T_n$ or continuous with $t \in [0, T_n]$. Let $\ell_n^t$ be càdlàg when $t$ is continuous. Then the above central limit condition holds for $[\ell^n, \ell^n]_{T_n}$ if and only if it holds for $(\ell^n, \ell^n)_{T_n}$. In this case, both $\psi_o$ and $\psi_p$ are well defined together and

$$(\psi_o - \psi_p)(z) = zb^{-3} \mu_3$$  \hspace{1cm} (4.1)

This result weakens the conditions required for Proposition 2 in Mykland (1993a).

Applying the above theorem to the results of Mykland (1993a) and extending to the continuous $t$ case, we have the following:

**Corollary 1.** Suppose that conditions (3.2) – (3.7) and the central limit condition above holds, and that $\psi_p$ and $\psi_o$ are differentiable, $\psi_p', \psi_o'$ are absolutely continuous and
\( \psi_p(z), \psi'_p(z), \psi_o(z)z, \psi'_o(z), \psi_s(z)z \) are \( o(\phi(z)^{-1}) \), then

\[
F_n(x) = \Phi(\beta^{-1}x) + r_n \frac{1}{2} \Lambda(\beta^{-1}x) \phi(\beta^{-1}x) + o_2(r_n),
\]  

(4.2)

where \( o_2 \) is as defined in Mykland (1992),

\[
\Lambda(z) = \psi'_p(z) - \psi_p(z)z + \frac{1}{3}(\mu_3 b^{-3})(1-z^2) + \psi_s(z)z
\]

\[
= \psi'_o(z) - \psi_o(z)z - \frac{2}{3}(\mu_3 b^{-3})(1-z^2) + \psi_s(z)z
\]  

(4.3)

and \( \beta = b_b^{-1} \) is the asymptotic standard deviation of \( \ell_{i,n}^n / \sigma_n \).

5. Proofs

The proofs are divided into two parts. First the discrete \( t \) case is considered and the results are proved to hold. Then the continuous \( t \) case is obtained as a generalization of the discrete \( t \) case.

5.1 The Discrete \( t \) Case

Proof of Theorem 1. To prove theorem 1 when \( t \) is discrete, we need the following lemma.

Lemma 1. Let \( g \) be a differentiable function such that \( g, g' \) are both continuous and bounded. Let \( \ell^n_t \) be a zero mean martingale array with \( t \) discrete, where \( t = 0, 1, 2, \ldots, T_n \). Let \( \ell^n_t \) be embedded in a (time- and samplepath-) continuous martingale \( \tilde{\ell}^n_t \) such that \( \tilde{\ell}^n_t = \ell^n_t \) for all integers \( i \). Suppose that condition (3.2) – (3.4) and (3.7) are satisfied and

\[
A = \inf \{ t : (\ell^n, \tilde{\ell}^{m})_t = b^2 c_n \}
\]  

(5.1)

then

\[
E g\left( \frac{\ell^n_{t_n}}{b\sqrt{c_n}} \right) r^{-1} c_n^{-1} \left[ (\ell^n, \tilde{\ell}^{m})_{t_n} - (\ell^n, \ell^n)_{t_n} \right] - E \mu_3 b^{-1} g'(\frac{\tilde{\ell}^{m}_{t_n}}{b\sqrt{c_n}}) \to 0
\]  

as \( n \to \infty \).

Proof of Lemma 1.

Set \( m^n_t = r^{-1} c_n^{-1} \left[ (\ell^n, \tilde{\ell}^{m})_t - (\ell^n, \ell^n)_t \right] \) and let \( m^n_t \) be embedded in continuous martingale \( m^n_t \). Also let \( X^n_t = \ell^n_t - \ell^n_{t-1} \) be the martingale difference.

Fact 1. \( E(m^n_{t_n})^2 \) is bounded.

Proof of Fact 1. By property of martingales and Jensen’s inequality,

\[
E(m^n_{t_n})^2 = r^{-2} c_n^{-2} E \sum_{i=1}^{T_n} \left[ (X^n_i)^2 - E((X^n_i)^2 | F^n_{i-1}) \right]^2
\]

\[
\leq 2r^{-2} c_n^{-2} E \sum_{i=1}^{T_n} \left[ (X^n_i)^4 + E((X^n_i)^4 | F^n_{i-1}) \right]
\]

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\[ = 4n^{-2} c_n^{-2} E \sum_{i=1}^{T_n} (X_i^n)^4 \]
\[ = 4n^{-2} c_n^{-2} E[\ell^n, \pi^n, \ell^n, \pi^n]_{T_n} \]
\[ = O(1) \quad \text{by (3.2).} \]

**FACT 2.** \( g \left( \frac{\ell_n}{b \sqrt{c_n}} \right) - g \left( \frac{\pi_n}{b \sqrt{c_n}} \right) \xrightarrow{p} 0 \)

**PROOF OF FACT 2.** Since \( g' \) is bounded, \( g \) is uniformly continuous. Therefore it suffices to show that

\[ \frac{\ell_n}{b \sqrt{c_n}} - \frac{\pi_n}{b \sqrt{c_n}} \xrightarrow{p} 0. \quad (5.3) \]

To proof this, let

\[ \tau = \inf \{ t : c_n^{-1}[\ell^n, \ell^n]_{t+1} > \bar{k} \} \wedge T_n \]

where the infimum of a empty set is defined as infinity, then

\[ P(\tau \neq T_n) \leq P(c_n^{-1}[\ell^n, \ell^n]_{\tau+1} > \bar{k} \text{ and } \tau < T_n) \]
\[ \leq P(c_n^{-1}[\ell^n, \ell^n]_{\tau} > \bar{k}) \quad \text{since } [\ell^n, \ell^n]_{t} \text{ is nondecreasing in } t \]
\[ = o(1) \quad \text{by (3.4).} \]

Hence

\[ P(\left| \frac{\ell_n}{b \sqrt{c_n}} - \frac{\pi_n}{b \sqrt{c_n}} \right| > \epsilon) \leq P \left( \left| \frac{\ell_n}{b \sqrt{c_n}} - \frac{\pi_n}{b \sqrt{c_n}} \right| > \epsilon \right) + o(1) \]
\[ \leq e^{-2} E \left( \frac{\ell_n}{b \sqrt{c_n}} - \frac{\pi_n}{b \sqrt{c_n}} \right)^2 + o(1) \]
\[ = e^{-2} b^{-2} c_n^{-1} E \left| \langle \ell^n, \ell^n \rangle_{\tau} - \langle \pi^n, \pi^n \rangle_A \right| + o(1) \]
\[ = e^{-2} b^{-2} c_n^{-1} E \left| \langle \ell^n, \ell^n \rangle_{\tau} - b^2 c_n \right| + o(1) \quad \text{by (5.1)} \]
\[ \leq e^{-2} b^{-2} c_n^{-1} E \left| \langle \ell^n, \ell^n \rangle_{\tau} - \langle \ell^n, \ell^n \rangle_{\tau} \right| \]
\[ + e^{-2} b^{-2} c_n^{-1} E \left| \langle \ell^n, \ell^n \rangle_{\tau} - b^2 c_n \right| + o(1) \]
\[ = e^{-2} b^{-2} c_n^{-1} E \left| \langle \ell^n, \ell^n \rangle_{\tau} - \langle \ell^n, \ell^n \rangle_{\tau} \right| + o(1) \]

since the second term is \( o(1) \) by (3.3) and (3.4) and \( P(\tau \neq T_n) = o(1) \)

\[ \leq e^{-2} b^{-2} c_n^{-1} \left\{ E \left[ (\ell^n, \ell^n)_{\tau} - \langle \ell^n, \ell^n \rangle_{\tau} \right]^2 \right\}^{1/2} + o(1) \]
\[ \leq 2e^{-2} b^{-2} c_n^{-1} \left\{ E \int_0^{T_n} (\tilde{\ell}_s^n - \pi_s^n) [\ell^n, \ell^n]_{s+1} \right\}^{1/2} + o(1) \]

by martingale property and Ito's formula

\[ = 2e^{-2} b^{-2} c_n^{-1} \left\{ E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n} \right\}^{1/2} + o(1) \]
\[ = o(1) \quad \text{by (3.2)} \]
since
\[
[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n} = 4 \int_0^{T_n} (\bar{\ell}^n_s - \bar{\ell}^{n}_{[s]} )^3 d\bar{\ell}^n_s + 6 \int_0^{T_n} (\bar{\ell}^n_s - \bar{\ell}^{n}_{[s]} )^2 d\langle \bar{\ell}^n, \bar{\ell}^n \rangle_s
\]  
(5.4)

This proves (5.3) and hence Fact 2.

FACT 3. \[ E\bar{m}^n_{T_n} g \left( \frac{\bar{m}^n}{b\sqrt{c_n}} \right) - E\bar{m}^n_{T_n} g \left( \frac{\bar{m}^n}{b\sqrt{c_n}} \right) \to 0 \] 
(5.5)

PROOF OF FACT 3.

First, note that \( m^n_{T_n} \) is the same as \( \bar{m}^n_{T_n} \). By Fact 1, \( m^n_{T_n} \) is uniformly integrable and hence tight. If Fact 3 does not hold, then there exist a \( \epsilon > 0 \) and a subsequence \( n_k \) such that the absolute value of the left hand side of (5.5) is greater than \( \epsilon \) for all elements of \( n_k \). By tightness, \( m^n_{T_n} \) converges in law through a further subsequence \( n_{k_i} \) of \( n_k \). By Fact 2, this leads to contradiction since Fact 3 will hold through \( n_{k_i} \) by the uniform integrability of \( m^n_{T_n} \) and the boundedness of \( g \).

FACT 4. \( b^{-1} c_n^{-1/2} \langle \bar{\ell}^n, \bar{m}^n \rangle_{A \wedge T_n} \) is uniformly integrable.

PROOF OF FACT 4. Using Kunita-Watanabe inequality,
\[
\langle \bar{\ell}^n, \bar{m}^n \rangle_{A \wedge T_n} \leq \langle \bar{m}^n, \bar{m}^n \rangle_{A \wedge T_n} \langle \bar{\ell}^n, \bar{\ell}^n \rangle_{A \wedge T_n} \\
\leq \langle \bar{m}^n, \bar{m}^n \rangle_{T_n} \langle \bar{\ell}^n, \bar{\ell}^n \rangle_A
\]

Hence,
\[
E \frac{\langle \bar{\ell}^n, \bar{m}^n \rangle^2}{b^2 c_n} \leq E \frac{\langle \bar{m}^n, \bar{m}^n \rangle_{T_n} \langle \bar{\ell}^n, \bar{\ell}^n \rangle_A}{b^2 c_n} = E(\bar{m}^n, \bar{m}^n)_{T_n} = O(1)
\]

since \( \langle \bar{\ell}^n, \bar{\ell}^n \rangle_A = b^2 c_n \) by Fact 1.

FACT 5.
\[
\frac{\langle \bar{\ell}^n, \bar{m}^n \rangle_{A \wedge T_n} - \langle \bar{\ell}^n, \bar{m}^n \rangle_{T_n}}{b\sqrt{c_n}} \overset{p}{\to} 0
\]

PROOF OF FACT 5. Since
\[
|\langle \bar{\ell}^n, \bar{m}^n \rangle_{T_n} - \langle \bar{\ell}^n, \bar{m}^n \rangle_{A \wedge T_n}| \\
\leq \sqrt{\langle \bar{m}^n, \bar{m}^n \rangle_{T_n} - \langle \bar{m}^n, \bar{m}^n \rangle_{A \wedge T_n}} \sqrt{\langle \bar{\ell}^n, \bar{\ell}^n \rangle_{T_n} - \langle \bar{\ell}^n, \bar{\ell}^n \rangle_{A \wedge T_n}} \\
\leq \sqrt{\langle \bar{m}^n, \bar{m}^n \rangle_{T_n}} \sqrt{\langle \bar{\ell}^n, \bar{\ell}^n \rangle_{T_n} - b^2 c_n}
\]

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hence,

\[
P \left( \frac{\left\langle \bar{\tilde{m}}^n, m^n \right\rangle_{A \wedge T_n} - \left\langle \tilde{m}^n, m^n \right\rangle_{T_n}}{b \sqrt{c_n}} > \varepsilon \right) \leq \frac{\left\langle \bar{\tilde{m}}^n, m^n \right\rangle_{A \wedge T_n} - \left\langle \tilde{m}^n, m^n \right\rangle_{T_n}}{b \sqrt{c_n}} \left\{ \kappa \geq \frac{(r_n c_n)_{T_{k}}}{c_n} \geq \bar{k} \right\} \geq \varepsilon \right) + o(r_n)
\]

by (3.4) for some \(\bar{k}, \bar{k}\)

\[
= \varepsilon^{-1}b^{-1}c_n^{-1/2}E \left| \left\langle \bar{\tilde{m}}^n, m^n \right\rangle_{A \wedge T_n} - \left\langle \tilde{m}^n, m^n \right\rangle_{T_n} \right| I \left\{ \kappa \geq \frac{(r_n c_n)_{T_{k}}}{c_n} \geq \bar{k} \right\} + o(r_n)
\]

\[
\leq \varepsilon^{-1}b^{-1}\sqrt{E(\tilde{m}^n, \tilde{m}^n)_{T_n}} \left\{ \kappa \geq \frac{(r_n c_n)_{T_{k}}}{c_n} \geq \bar{k} \right\} + o(r_n)
\]

\[= o(1) \quad \text{by fact 1 and (3.3).}
\]

**FACT 6.**

\[
\frac{[\ell^n, m^n]_{T_n}}{b \sqrt{c_n}} - \frac{\langle \bar{\tilde{m}}^n, m^n \rangle_{T_n}}{b \sqrt{c_n}} \xrightarrow{P} 0
\]

**PROOF OF FACT 6.** By Ito’s formula,

\[
\frac{[\ell^n, m^n]_{T_n}}{b \sqrt{c_n}} - \frac{\langle \bar{\tilde{m}}^n, m^n \rangle_{T_n}}{b \sqrt{c_n}}
\]

\[
= b^{-1}c_n^{-1/2} \int_0^{T_n} (\bar{\tilde{m}}^n_s - m^n_s) d\tilde{e}^n_s + b^{-1}c_n^{-1/2} \int_0^{T_n} (\bar{\tilde{m}}^n_s - \tilde{m}^n_s) d\bar{\tilde{m}}^n_s
\]

It will be shown that both integrals on the right hand side of the above goes to 0 in probability.

For the first term, let

\[
\sigma = \inf\{t : |\langle \bar{\tilde{e}}^n, \tilde{e}^n \rangle_t - \langle \bar{\tilde{e}}^n, \tilde{e}^n \rangle_{[t]}| \geq \epsilon_n c_n \} \wedge T_n
\]

where \(\epsilon_n = (c_n^{-2}E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n})^{1/4} = o(1)\) by (3.2)

then

\[
P(\sigma \neq T_n) = o(1)
\]

because

\[
P(\sigma \neq T_n) \leq P \left( |\langle \bar{\tilde{e}}^n, \tilde{e}^n \rangle_{\sigma} - \langle \bar{\tilde{e}}^n, \tilde{e}^n \rangle_{[\sigma]}| \geq \epsilon_n c_n \right)
\]

\[
\leq P \left( \sum_{i=1}^{T_n} [\langle \bar{\tilde{e}}^n, \tilde{e}^n \rangle_i - \langle \bar{\tilde{e}}^n, \tilde{e}^n \rangle_{i-1}]^2 \geq \epsilon_n^2 c_n^2 \right)
\]
since \((\tilde{\bar{e}}^n, \tilde{\bar{e}}^n)\) is nondecreasing in \(t\)

\[
\leq P\left( \sum_{i=1}^{T_n} (X^n_i)^4 + ((X^n_i)^2 - (\tilde{\bar{e}}^n, \tilde{\bar{e}}^n) - (\tilde{\bar{e}}^n_{i-1}))^2 \geq \frac{\epsilon_n^2 c_n^2}{4} \right)
\]

by the inequality \(2a^2 + 2(a - b)^2 \geq b^2\)

\[
\leq P\left( \sum_{i=1}^{T_n} (X^n_i)^4 \geq \frac{\epsilon_n^2 c_n^2}{4} \right) + P\left( \sum_{i=1}^{T_n} ((X^n_i)^2 - (\tilde{\bar{e}}^n, \tilde{\bar{e}}^n) - (\tilde{\bar{e}}^n_{i-1}))^2 \geq \frac{\epsilon_n^2 c_n^2}{4} \right)
\]

\[
\leq 4\epsilon_n^{-2} c_n^{-2} E[\tilde{\bar{e}}^n, \ell^n, \tilde{\bar{e}}^n, \ell^n] T_n + 4\epsilon_n^{-2} c_n^{-2} E \sum_{i=1}^{T_n} ((X^n_i)^2 - (\tilde{\bar{e}}^n, \tilde{\bar{e}}^n) - (\tilde{\bar{e}}^n_{i-1}))^2
\]

\[
= 4\epsilon_n^2 + 4\epsilon_n^{-2} c_n^{-2} E \sum_{i=1}^{T_n} \left( \int_{t_{i-1}}^t 2(\tilde{\bar{e}}^n_s - \tilde{\bar{e}}^n_{t_{i-1}}) d\tilde{\bar{e}}^n_s \right)^2
\]

by Ito’s formula

\[
= 4\epsilon_n^2 + 16\epsilon_n^{-2} c_n^{-2} E \left[ \int_0^{T_n} (\tilde{\bar{e}}^n_s - \tilde{\bar{e}}^n_{t_{i-1}}) d\tilde{\bar{e}}^n_s \right]^2
\]

by martingale property

\[
= 4\epsilon_n^2 + 16\epsilon_n^{-2} c_n^{-2} E \int_0^{T_n} (\tilde{\bar{e}}^n_s - \tilde{\bar{e}}^n_{t_{i-1}})^2 d\langle \tilde{\bar{e}}^n, \tilde{\bar{e}}^n \rangle_s
\]

\[
= 4\epsilon_n^2 + 16\epsilon_n^{-2} c_n^{-2} E \frac{\tilde{\bar{e}}^n, \ell^n, \tilde{\bar{e}}^n, \ell^n T_n}{6}
\]

by (5.4)

\[
= o(1).
\]

Therefore,

\[
P\left( b^{-1} c_n^{-1/2} \int_0^{T_n} (\tilde{\bar{m}}^n_s - \tilde{\bar{m}}^n_{t_{i-1}}) d\tilde{\bar{e}}^n_s \mid > \epsilon \right)
\]

\[
= P\left( b^{-1} c_n^{-1/2} \int_0^{T_n} (\tilde{\bar{m}}^n_s - \tilde{\bar{m}}^n_{t_{i-1}}) d\tilde{\bar{e}}^n_s \mid > \epsilon \right) + o(1)
\]

\[
\leq \epsilon^{-2} b^{-2} c_n^{-1} E \int_0^{T_n} (\tilde{\bar{m}}^n_s - \tilde{\bar{m}}^n_{t_{i-1}})^2 d\langle \tilde{\bar{e}}^n, \tilde{\bar{e}}^n \rangle_s + o(1)
\]

by an application of Ito’s formula

\[
\leq \epsilon^{-2} b^{-2} c_n^{-1} E \sum_{i=1}^{T_n} (m^n_i - m^n_{i-1})^2 (\tilde{\bar{e}}^n, \tilde{\bar{e}}^n)_{t_{i-1}} \tilde{\bar{e}}^n_{t_{i-1}} - \tilde{\bar{e}}^n_{t_{i-1}})_{t_{i-1}} + o(1)
\]

since \((\tilde{\bar{e}}^n, \tilde{\bar{e}}^n)_{t_{i-1}} - \tilde{\bar{e}}^n_{t_{i-1}})_{t_{i-1}} \leq \epsilon_n c_n

by Fact 1.
For the second term, let

\[ \tau = \inf \{ t : | \ell^n_t - \ell^n_{[t]} | \geq \delta_n c_n^{1/2} \} \wedge T_n \]

where \( \delta_n = \{ c_n^2 E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n} \}^{1/8} = o(1) \) by (3.2)

then

\[
P(\tau \neq T_n) \leq P \left( | \ell^n_{[\tau]} - \ell^n_{[\tau]} | \geq \delta_n c_n^{1/2} \right)
\]
\[
\leq c_n^{-2} \delta_n^{-4} E(\ell^n_{[\tau]} - \ell^n_{[\tau]})^4
\]
\[
\leq c_n^{-2} \delta_n^{-4} E \{ E(\ell^n_{[\tau]} - \ell^n_{[\tau]} | | T^n_{\tau} ) \}^4
\]
\[
\quad \text{since } | \ell^n_{[\tau]} - \ell^n_{[\tau]} | \leq E(\ell^n_{[\tau]} - \ell^n_{[\tau]} | | T^n_{\tau} )
\]
\[
\leq c_n^{-2} \delta_n^{-4} E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n}
\]
\[
= \delta_n^4
\]
\[
= o(1) .
\]

Hence

\[
P \left( | b^{-1} c_n^{-1/2} \int_0^{T_n} (\ell^n_s - \ell^n_{[s]} ) d\bar{m}^n_s | > \epsilon \right)
\]
\[
= P \left( | b^{-1} c_n^{-1/2} \int_0^\tau (\ell^n_s - \ell^n_{[s]} ) d\bar{m}^n_s | > \epsilon \right) + o(1)
\]
\[
\leq \epsilon^{-2} b^{-2} c_n^{-1} E \int_0^\tau (\ell^n_s - \ell^n_{[s]} )^2 d\langle m^n, m^n \rangle_s + o(1)
\]
\[
\leq \epsilon^{-2} b^{-2} \delta_n^2 E \int_0^\tau d\langle m^n, m^n \rangle_s + o(1)
\]
\[
\leq \epsilon^{-2} b^{-2} \delta_n^2 E \langle m^n, m^n \rangle_{T_n} + o(1)
\]
\[
= o(1)
\]

by Fact 1.

FACT 7. \[ \frac{[\ell^n, m^n]_{T_n}}{b \sqrt{c_n}} - \frac{[\ell^n, f, f]_{T_n}}{b \tau_n c_n^{3/2}} P \to 0 \]

PROOF OF FACT 7. Let

\[ \tau = \inf \{ t : | X^n_{t+1} | > \sqrt{c_n \epsilon_n} \} \wedge T_n \]

where \( \epsilon_n = c_n^{-1/4} (E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n})^{1/8} = o(1) \)

then

\[
P(\tau \neq T_n) \leq P( | X^n_{\tau+1} | > \sqrt{c_n \epsilon_n} \text{ and } \tau < T_n)
\]
\[
\leq c_n^{-2} \epsilon_n^{-4} E[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n}
\]
\[
= \epsilon_n^4
\]
\[
= o(1) .
\]
Note that
\[
\frac{[\bar{\ell}^n, \ell^n, \bar{\ell}^n]_{T_n}}{b \, c_n^{3/2}} \rightarrow \frac{[\bar{\ell}^n, m^n]_{T_n}}{b \sqrt{c_n}} = b^{-1} r_n^{-1} c_n^{-3/2} \sum_{i=1}^{T_n} X_i^n \mathbb{E}(X_i^n)^2 \mid \mathcal{F}^n_{i-1}
\]
is a martingale, therefore
\[
P \left( \frac{[\bar{\ell}^n, \ell^n, \bar{\ell}^n]_{T_n}}{b \, r_n c_n^{3/2}} \mid \mathcal{F}^n_{i-1} > \epsilon \right)
\]
\[
= P \left( \frac{1}{b^{-1} r_n^{-1} c_n^{-3/2}} \sum_{i=1}^{\bar{r}} X_i^n \mathbb{E}(X_i^n)^2 \mid \mathcal{F}^n_{i-1} > \epsilon \right) + o(1)
\]
\[
\leq \epsilon^{-2} b^{-2} r_n^{-2} c_n^{-3} \mathbb{E}\left\{ \sum_{i=1}^{\bar{r}} X_i^n \mathbb{E}(X_i^n)^2 \mid \mathcal{F}^n_{i-1} \right\} + o(1)
\]
\[
= \epsilon^{-2} b^{-2} r_n^{-2} c_n^{-3} \mathbb{E}\left\{ \sum_{i=1}^{\bar{r}} X_i^n \mathbb{E}(X_i^n)^2 \mid \mathcal{F}^n_{i-1} \right\} + o(1)
\]
by martingale property
\[
\leq \epsilon^{-2} b^{-2} r_n^{-2} c_n^{-2} \epsilon_n^2 \mathbb{E}\left\{ \sum_{i=1}^{\bar{r}} \mathbb{E}(X_i^n)^2 \mid \mathcal{F}^n_{i-1} \right\} + o(1)
\]
since $X_i^n \leq \sqrt{c_n} \epsilon_n$ for $i \leq \bar{r}$
\[
\leq \epsilon^{-2} b^{-2} r_n^{-2} c_n^{-2} \epsilon_n^2 \mathbb{E}\left\{ [\bar{\ell}^n, \ell^n, \bar{\ell}^n]_{T_n} \right\} + o(1)
\]
\[
= \epsilon^{-2} b^{-2} c_n^2 + o(1)
\]
\[
= o(1) .
\]

Now by equation (7.4) of Theorem 6 in Mykland (1993b)
\[
E m_{T_n}^{\tilde{m}_n} \left( \frac{\tilde{m}_A^n}{b \sqrt{c_n}} \right) = E g' \left( \frac{\tilde{m}_A^n}{b \sqrt{c_n}} \right) \frac{\langle \tilde{m}_n, \tilde{m}_n \rangle_{A \wedge T_n}}{b \sqrt{c_n}} .
\]

Combining this with Fact 3 gives
\[
E m_{T_n}^{\tilde{m}_n} \left( \frac{\tilde{m}_A^n}{b \sqrt{c_n}} \right) - E g' \left( \frac{\tilde{m}_A^n}{b \sqrt{c_n}} \right) \frac{\langle \tilde{m}_n, \tilde{m}_n \rangle_{A \wedge T_n}}{b \sqrt{c_n}} \rightarrow 0 .
\] (5.6)

By Fact 5, Fact 6 and Fact 7,
\[
\frac{\langle \tilde{m}_n, \tilde{m}_n \rangle_{A \wedge T_n}}{b \sqrt{c_n}} - \frac{[\tilde{m}_n, \ell^n, \ell^n]_{T_n}}{b \, r_n c_n^{3/2}} \rightarrow 0 .
\]

With condition (3.7), this gives
\[
\frac{\langle \tilde{m}_n, \tilde{m}_n \rangle_{A \wedge T_n}}{b \sqrt{c_n}} - b^{-1} \mu_3 \rightarrow 0 .
\]
Since \( g' \) is bounded and by Fact 4, \( b^{-1} c_n^{1/2} \left( \bar{\ell}_n, \bar{n}_n \right)_{A \land T_n} \) is uniformly integrable, therefore

\[
E g' \left( \frac{\bar{\ell}_n}{b \sqrt{c_n}} \right) \left( \frac{\bar{\ell}_n, \bar{n}_n}{b \sqrt{c_n}} \right)_{A \land T_n} - E \frac{b}{\sqrt{c_n}} \left( \frac{\bar{\ell}_n}{b \sqrt{c_n}} \right) \rightarrow 0 .
\]

Putting this into (5.6) gives Lemma 1.

To prove Theorem 1, suppose that the Central limit condition holds for \( (\ell^n, \ell^n)_{T_n} \) and hence that \( \psi_p \) is well defined.

Let \( n_k \) be a subsequence of integers for which

\[
\left( b^{-1} \frac{\ell_n}{\sqrt{c_n}}, \left( \left( [\ell_n, \ell^n]_{T_n} - b^2 \right) \right) r_n^{-1} \left( \frac{\sigma_n^2}{c_n} - b_*^2 \right) \right) \xi_n (Z, \eta, \xi) .
\]

By (3.3) and (3.4) \( r_n^{-1} \left( \frac{[\ell_n, \ell^n]_{T_n}}{c_n} - b^2 \right) \) is tight, therefore there exist a further subsequence \( n_{k_j} \) of \( n_k \) through which

\[
\left( b^{-1} \frac{\ell_n}{\sqrt{c_n}}, \left( \left( [\ell_n, \ell^n]_{T_n} - b^2 \right) \right) r_n^{-1} \left( \frac{\sigma_n^2}{c_n} - b_*^2 \right) \right)
\]

converges in law to \( (Z, \eta, \xi, \xi_p) \).

By the uniform integrability of \( m_{T_n}^n \) which follows from Fact 1, for \( g \) such that \( g, g' \) are both continuous and bounded,

\[
E g \left( \frac{\ell_n}{b \sqrt{c_n}} \right) r_n^{-1} \left( [\ell_n, \ell^n]_{T_n} - \langle \ell_n, \ell^n \rangle_{T_n} \right) = E g \left( \frac{\ell_n}{b \sqrt{c_n}} \right) m_{T_n}^n \\
\overset{n_{k_j}}{\rightarrow} E g(Z)(\xi - \xi_p) .
\]

By (5.3), \( \bar{\ell}_n \) converges in law through \( n_{k_j} \) to \( Z \) also, hence

\[
E b^{-1} \mu_3 g' \left( \frac{\bar{\ell}_n}{b \sqrt{c_n}} \right) \overset{n_{k_j}}{\rightarrow} E b^{-1} \mu_3 g'(Z) = E b^{-1} \mu_3 g(Z) Z
\]

using integration by parts. Therefore, putting this and (5.7) into (5.2) of Lemma 1, we have

\[
E g(Z)(\xi - \xi_p) = E b^{-1} \mu_3 g(Z) Z .
\]

If \( \psi_o \) is equal to a version of \( b^{-2} E (\xi_o \mid Z) \) this leads to

\[
E g(Z) (\psi_o - \psi_p - b^{-3} \mu_3 Z) = 0 .
\]

Since, this holds for all \( g \) such that \( g, g' \) is bounded and continuous, we must have

\[
(\psi_o - \psi_p)(z) = z b^{-3} \mu_3 .
\]
Further, $\psi_p$ and $zb^{-3}\mu_3$ are independent of the $n_k$, hence, $\psi_o$ is independent of $n_k$ also.

Since the argument can be reversed between the optional and the predictable variation, this proves Theorem 1.

**Proof of Corollary 1.**

In the discrete $t$ case, our conditions satisfy all those of Mykland (1993a). Using (4.1) and the results of Mykland (1993a), (4.2) and (4.3) can then be derived.

*5.2 The Continuous Case*

The proofs given here for extending the discrete $t$ case to the continuous $t$ case make use of the methods of Mykland (1990). The setup is as follows.

For some sequence $\epsilon_n > 0$, which is chosen by conditions later, create the following stopping times:

$$\tau_0^n = 0$$

$$\tau_{u+1}^n = \begin{cases} 
(\tau_u^n + \epsilon_n) \land T_n & \text{if } |\ell_s^n - \ell_{\tau_u^n}^n| \leq 1 \text{ for all } s \text{ such that } \tau_u^n \leq s \leq (\tau_u^n + \epsilon_n) \land T_n \\
\inf\{s : |\ell_s^n - \ell_{\tau_u^n}^n| > 1 \text{ for some } i\} & \text{otherwise}
\end{cases}$$

Then by the arguments in Mykland (1990), there exist a $M_n$ such that $\tau_{M_n} = T_n$ a.s.

If $\ell_t^n$ is now a continuous $t$ martingale, then a discrete $t$ martingale $\tilde{\ell}_t^n$, with $i = 0, \ldots, M_n$, can be constructed by

$$\tilde{\ell}_t^n = \ell_{\tau_t^n}.$$ 

Now, suppose that conditions (3.2)-(3.4) (3.7) and the central limit condition are satisfy by $\ell_t^n$. Since $\tilde{\ell}_{M_n} = \ell_T$ a.s., it is enough to prove theorem 1 and corollary 1 if we check the conditions on the variation processes associated with $\tilde{\ell}_t^n$.

By the results of Mykland (1990), (3.2)-(3.4) is satisfied by $\tilde{\ell}_t^n$, and $\psi_o$ exists for $\tilde{\ell}_t^n$ in the same form as for $\ell_t^n$. Similarly, $\psi_p$ has the same form for $\tilde{\ell}_t^n$ and $\ell_t^n$ and exists for $\tilde{\ell}_t^n$ and $\ell_t^n$ together. Hence, it remains to show that condition (3.7) for the third order variation is satisfied by $\tilde{\ell}_t^n$.

By applying Theorem 1.4.47 (p.52) of Jacod & Shiryaev (1987), for each fixed $n$,

$$[\tilde{\ell}^n, \tilde{\ell}^n, \tilde{\ell}^n]_{M_n} \overset{P}{\to} [\ell^n, \ell^n, \ell^n]_{T_n}$$

as $\epsilon_n$ goes to zero. Further this convergence is dominated since

$$[\tilde{\ell}^n, \tilde{\ell}^n, \tilde{\ell}^n]_{M_n} \leq [\tilde{\ell}^n, \tilde{\ell}^n]_{M_n} + [\tilde{\ell}^n, \tilde{\ell}^n, \tilde{\ell}^n, \tilde{\ell}^n]_{M_n}$$

by the algebraic inequality $x^3 \leq x^2 + x^4$ and

$$E[\tilde{\ell}^n, \tilde{\ell}^n]_{M_n} = E[\ell^n, \ell^n]_{T_n} < \infty$$

and $E[\tilde{\ell}^n, \tilde{\ell}^n, \tilde{\ell}^n]_{M_n}$ is dominated by (6.4) of Mykland (1990). Since dominated convergence in probability implies convergence in $L_1$, therefore $\epsilon_n$ can be chosen so that for each $n$,

$$E |[\tilde{\ell}^n, \tilde{\ell}^n, \tilde{\ell}^n]_{M_n} - [\ell^n, \ell^n, \ell^n]_{T_n}| = o(\tau_n)$$

(5.9)
Now, for this choice of $\epsilon_n$, if $n$ is let to go to infinity, then
\[
[\tilde{\epsilon}^n, \epsilon^n, \tilde{\epsilon}^n]_{M_n} \overset{p}{\to} [\epsilon^n, \epsilon^n, \epsilon^n]_{T_n}
\]
as $n \to \infty$. Hence, condition (3.7) for the third order variation is satisfied by $\tilde{\epsilon}^n$ if and only if it is satisfied by $\epsilon^n$. This completes the proofs for the continuous $t$ case.
REFERENCES


Department of Biostatistics
University of Wisconsin
Madison, Wisconsin 53792

Department of Statistics
University of Chicago
5734 University Avenue
Chicago, Illinois 60637