# ANOVA, still

$\{Y_{ti}\}$ independent with $Y_{ti} \sim \text{normal}(\mu_t, \sigma)$ for t = 1 . . . k.

Test $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$

The usual statistic:

$$F = M_B/M_W = \frac{\sum_t n_t\,(\bar{Y}_{t\cdot} - \bar{Y}_{\cdot\cdot})^2/k}{\sum_t \sum_i (Y_{ti} - \bar{Y}_{t\cdot})^2/(\sum_t n_t - k)}$$

P-values: (a) Use the F(k, $\sum n_t - k$) distribution.

(b) Use a permutation test.

Assumptions: (a) Underlying dist'ns are normal with common SD.

(b) Underlying dist'ns are the same.

# Non-parametric ANOVA

An alternative approach: the Kruskal-Wallis test.

Rank all of the observations from 1, 2, . . . , N.

Let $R_{ti}$ = the rank for observation $Y_{ti}$.

Let $\bar{R}_{t\cdot} = \sum_i R_{ti}/n_t$ = the average rank for group t.

Null hypothesis, $H_0$: the underlying distributions are all the same.

$$E(\bar{R}_{t\cdot} \mid H_0) = \frac{N+1}{2}$$

$$SD(\bar{R}_{t\cdot} \mid H_0) = \sqrt{\frac{(N+1)\ (N-n_t)}{12\ n_t}}$$

# Kruskal-Wallis test statistic

$$H = \sum_t \left(\frac{N - n_t}{N}\right) \times \left[\frac{\bar{R}_{t\cdot} - E(\bar{R}_{t\cdot} \mid H_0)}{SD(\bar{R}_{t\cdot} \mid H_0)}\right]^2$$

$$= \cdots = \frac{12}{N\,(N+1)} \sum_t n_t \left[\bar{R}_{t\cdot} - \left(\frac{N+1}{2}\right)\right]^2$$

Under $H_0$, and if the sample sizes are large, $H \sim \chi^2(df = k - 1)$.

Alternatively, we could use a permutation test to estimate a P-value.

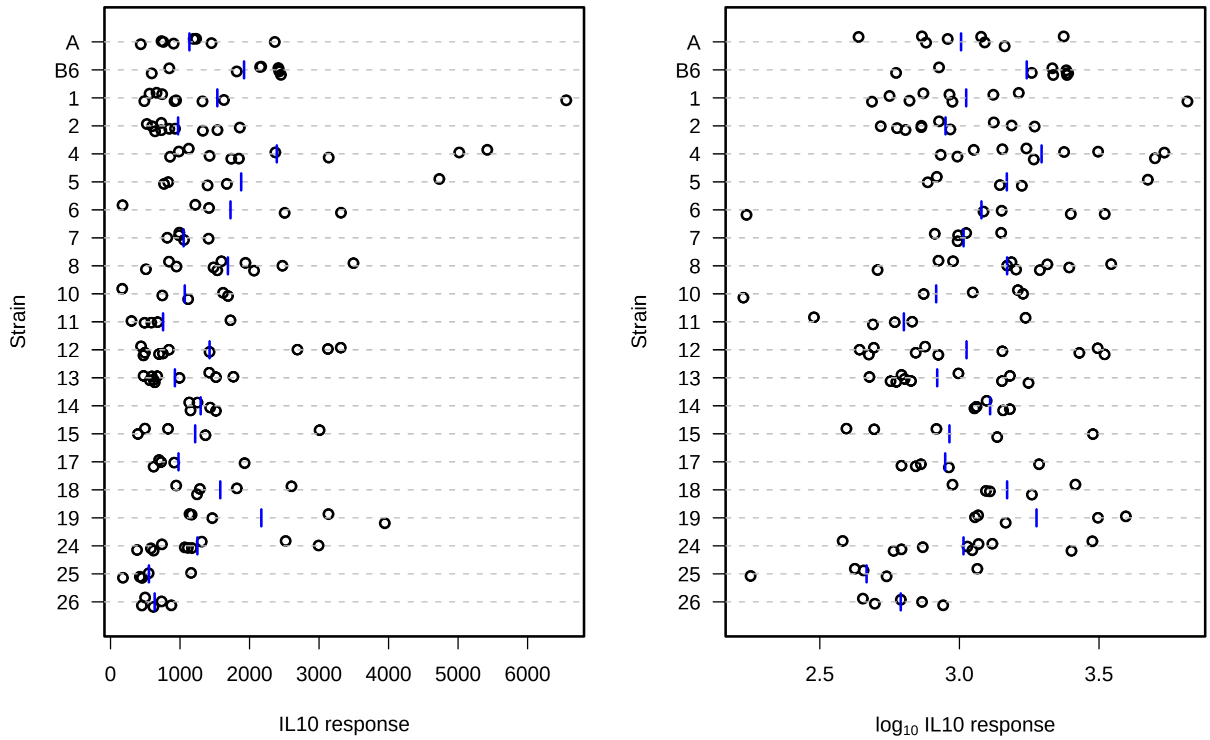
The function `kruskal.test()` in R will calculate the statistic.

# Note

In the case of two groups, the Kruskal-Wallis test reduces exactly to the Wilcoxon rank-sum test.

This is just like how ANOVA is equivalent to the two-sample t test.

# Example




# ANOVA Tables

## Original scale / 1000:

| source | SS | df | MS | F | P-value |
|---|---|---|---|---|---|
| between strains | 33 | 20 | 1.69 | 1.70 | 0.042 |
| within strains | 124 | 125 | 0.99 | | |
| total | 157 | 145 | | | |

permutation P-value = 0.043

## $\log_{10}$ scale:

| source | SS | df | MS | F | P |
|---|---|---|---|---|---|
| between strains | 3.35 | 20 | 0.167 | 2.25 | 0.0036 |
| within strains | 9.29 | 125 | 0.074 | | |
| total | 12.63 | 145 | | | |

permutation P-value = 0.003

# K-W results

The observed Kruskal-Wallis statistic for these data was 41.32. (Note that it doesn't matter whether you take logs.)

Since there were 21 strains, we can compare this to a $\chi^2$ distribution with 20 degrees of freedom. Thus we obtain the P-value = 0.003.

With a permutation test, I got $\hat{P}$ = 0.0015 (on the basis of 10,000 simulations.

# In the case of ties...

In the case of ties, we assign the average rank to each.

Example:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **A:** | | 3.5 | | 3.7 | | 4.0 | 4.2 | 4.3 | |
| **B:** | | | | | 3.9 | | | 4.3 | 4.5 |
| **C:** | 3.1 | | 3.6 | | | 4.0 | | 4.3 | |
| | (1) | (2) | (3) | (4) | (5) | (6/7) | (8) | (9/10/11) | (12) |
| | | | | | | ↓ | | ↓ | |
| | | | | | | 6.5 | | 10 | |

Then we apply a correction factor.

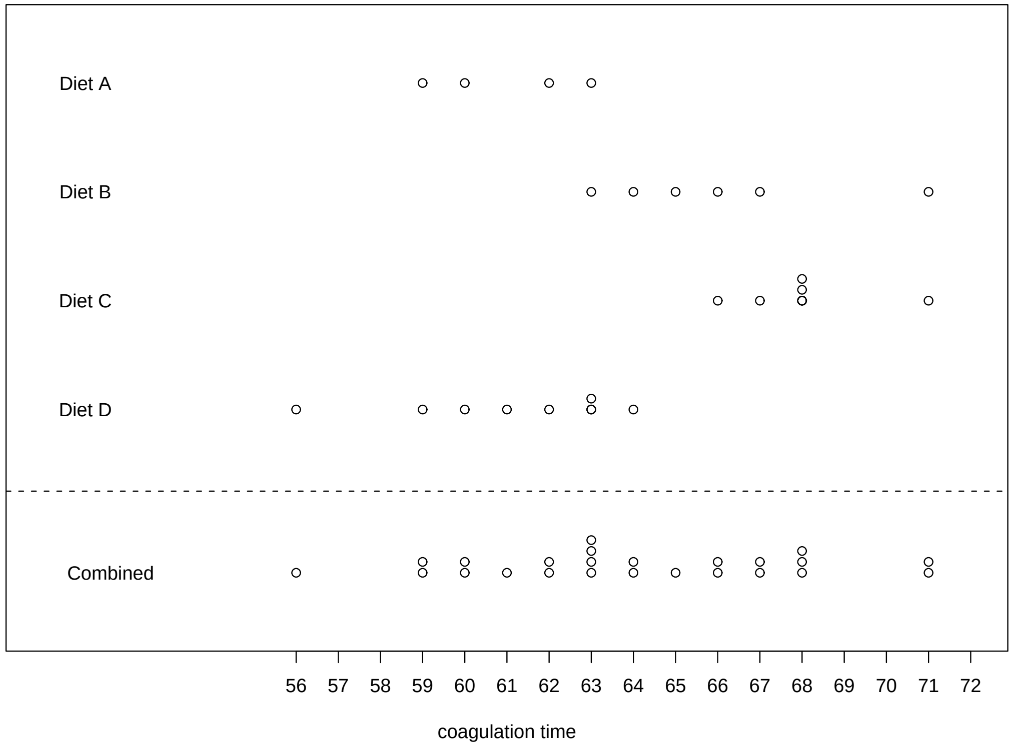
Let N = $\sum_t n_t$ and $T_i$ = no. observations in the ith set of ties (can be 1).

Let D = $1 - \sum_i (T_i^3 - T_i)/(N^3 - N)$

Use the statistic $H' = H/D$.

Note that $D \leq 1$ and so $H' \geq H$. For the example, $D = 1 - \frac{(2^3-2)+(3^3-3)}{12^3-12} \approx 0.983$.

# Blood coagulation time




# Example (continued)

| A | B | C | D | rank | avg rank |
|---|---|---|---|---|---|
| | | | 56 | 1 | 1 |
| 59 | | | | 2 | 2.5 |
| | | | 59 | 3 | 2.5 |
| 60 | | | | 4 | 4.5 |
| | | | 60 | 5 | 4.5 |
| | | | 61 | 6 | 6 |
| 62 | | | | 7 | 7.5 |
| | | | 62 | 8 | 7.5 |
| 63 | | | | 9 | 10.5 |
| | 63 | | | 10 | 10.5 |
| | | | 63 | 11 | 10.5 |
| | | | 63 | 12 | 10.5 |
| | 64 | | | 13 | 13.5 |
| | | | 64 | 14 | 13.5 |
| | 65 | | | 15 | 15 |
| | 66 | | | 16 | 16.5 |
| | | 66 | | 17 | 16.5 |
| | 67 | | | 18 | 18.5 |
| | | 67 | | 19 | 18.5 |
| | | 68 | | 20 | 21 |
| | | 68 | | 21 | 21 |
| | | 68 | | 22 | 21 |
| | 71 | | | 23 | 23.5 |
| | | 71 | | 24 | 23.5 |

# Example (continued)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | 62 | 60 | 63 | 59 | | | | | 61 |
| | 7.5 | 4.5 | 10.5 | 2.5 | | | | | 6.25 |
| B | 63 | 67 | 71 | 64 | 65 | 66 | | | 66 |
| | 10.5 | 18.5 | 23.5 | 13.5 | 15.0 | 16.5 | | | 16.25 |
| C | 68 | 66 | 71 | 67 | 68 | 68 | | | 68 |
| | 21.0 | 16.5 | 23.5 | 18.5 | 21.0 | 21.0 | | | 20.25 |
| D | 56 | 62 | 60 | 61 | 63 | 64 | 63 | 59 | 61 |
| | 1.0 | 7.5 | 4.5 | 6.0 | 10.5 | 13.5 | 10.5 | 2.5 | 7.00 |

# Calculation of K-W test statistic

| | A | B | C | D |
|---|---|---|---|---|
| $n_t$ | 4 | 6 | 6 | 8 |
| $\bar{R}_{t\cdot}$ | 6.25 | 16.25 | 20.25 | 7.00 |

N = 24

$\frac{N+1}{2} = 12.5$

$$\begin{aligned} H &= \frac{12}{N\,(N+1)} \sum_t n_t \left[\bar{R}_{t\cdot} - \left(\tfrac{N+1}{2}\right)\right]^2 \\ &= \frac{12}{24 \times 25} \left\{4 \times (6.25 - 12.5)^2 + \cdots + 8 \times (7.00 - 12.5)^2\right\} \\ &= 16.86 \end{aligned}$$

The ties: $T_i$ = ( 1 2 2 1 2 4 2 1 2 2 3 2 )

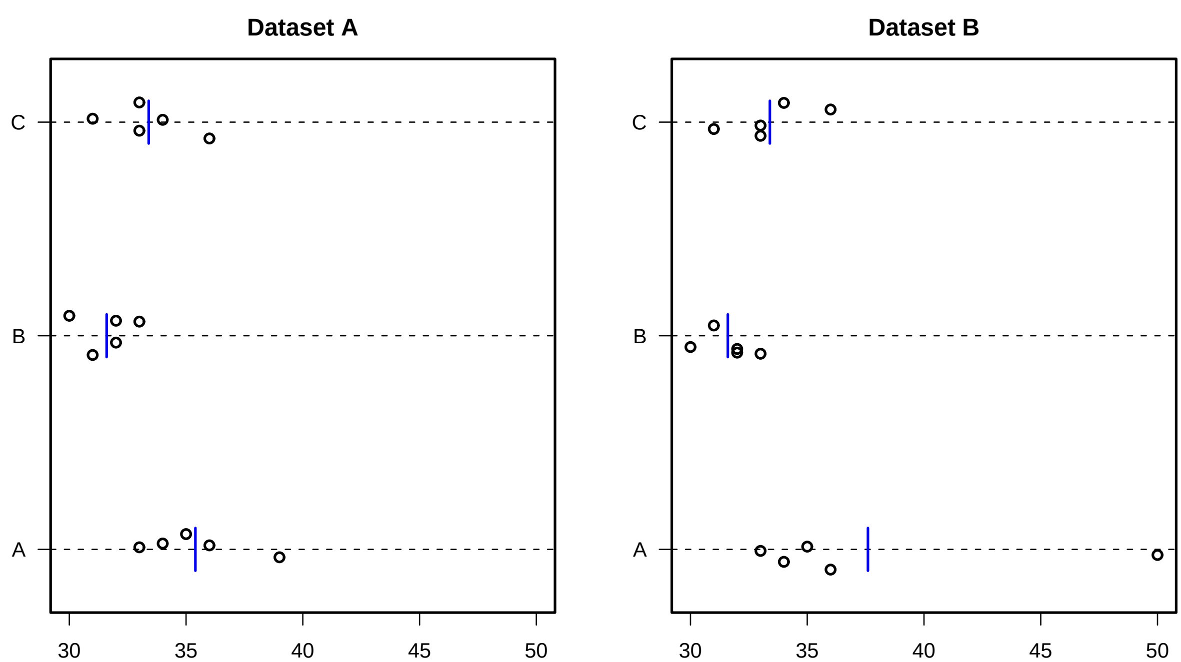
$D = 1 - \sum_i (T_i^3 - T_i)/(N^3 - N) = \ldots = 0.991$

$H' = H/D$ = 16.86 / 0.991 = 17.02 [df = 3] P-value $\approx$ 0.0007

# A few points

- Calculation of P-values: (avoiding type I errors)
  - F statistic: F distribution (requires normality)
  - K-W statistic: $\chi^2$ distribution (requires large samples)
  - Either statistic: Permutation tests
- Power: (avoiding type II errors)
  - K-W statistic more resistant to outliers
  - F statistic more powerful in the case of normality
- K-W statistic: don't need to worry about transformations.

# A fake example



# Results

| Dataset | Method | Statistic | nominal P-value | Permu'n P-value |
|---|---|---|---|---|
| **A** | ANOVA | 5.48 | 0.020 | 0.017 |
| | K-W | 7.64 | 0.022 | 0.012 |
| **B** | ANOVA | 2.64 | 0.112 | 0.023 |
| | K-W | 7.64 | 0.022 | 0.012 |

# Distributions