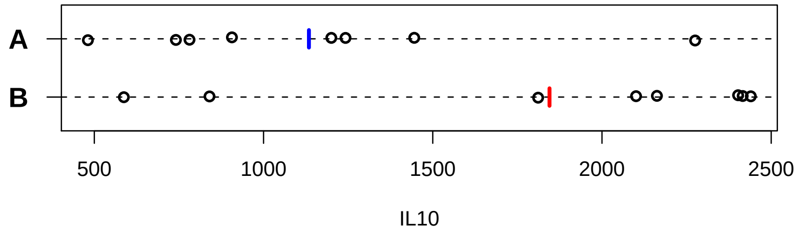
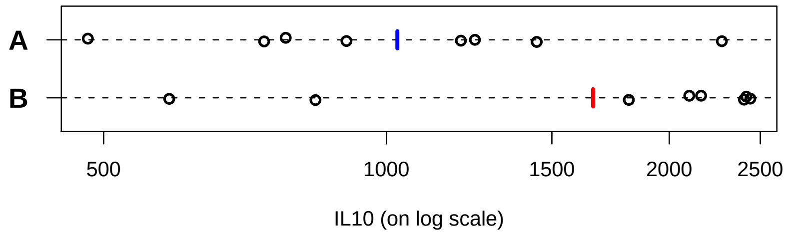
Example

Two strains of mice: A and B.
Measure cytokine IL10 (in males all the same age) after treatment.





Point: We're not interested in these particular mice, but in aspects of the distributions of IL10 values in the two strains.

Populations and samples

We are interested in the distribution of measurements in the underlying (possibly hypothetical) population.

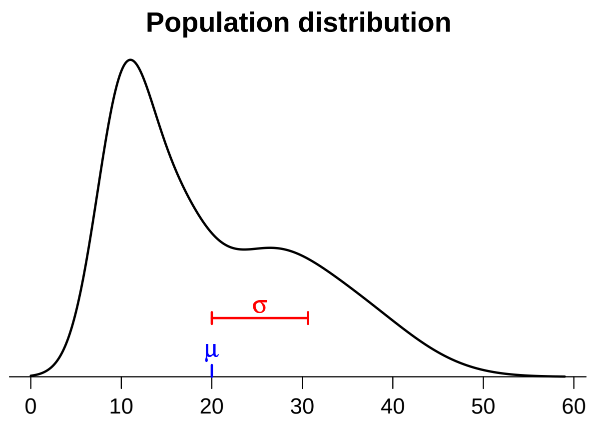
Examples:

- Infinite number of mice from strain A; cytokine response to treatment.
- All T cells in a person; respond or not to an antigen.
- All possible samples from the Baltimore water supply; concentration of cryptospiridium.
- All possible samples of a particular type of cancer tissue; expression of a certain gene.

We can't see the entire population (whether it is real or hypothetical), but we can see a random sample of the population (perhaps a set of independent, replicated measurements).

Parameters

The object of our interest is the population distribution or, in particular, certain numerical attributes of the population distribution (called parameters).



Examples:

- mean
- median
- SD
- proportion = 1
- proportion $>$ 40
- geometric mean
- 95th percentile

Parameters are usually assigned greek letters (like θ, μ, and σ).

Sample data

We make *n* independent measurements (or draw a random sample of size *n*).

This gives $X_1, X_2, \ldots, X_n$ independent and identically distributed (iid), following the population distribution.

Statistic: A numerical summary (function) of the X's (that is, of the data). For example, the sample mean, sample SD, etc.

Estimator: (estimate) A statistic, viewed as estimating some population parameter.

We write:

$\hat{\theta}$ an estimator of θ $\qquad$ $\bar{X} = \hat{\mu}$ an estimator of μ

$\hat{p}$ an estimator of p $\qquad$ $S = \hat{\sigma}$ an estimator of σ

Parameters, estimators, estimates

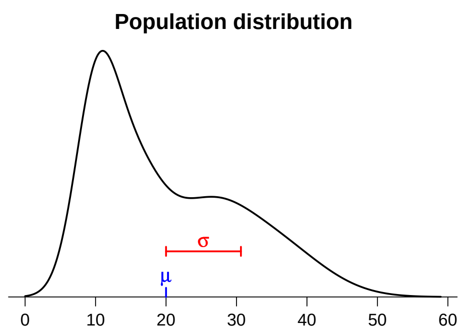
μ
- The population mean
- A parameter
- A fixed quantity
- Unknown, but what we want to know

$\bar{X}$
- The sample mean
- An estimator of μ
- A function of the data (the X's)
- A random quantity

$\bar{x}$
- The observed sample mean
- An estimate of μ
- A particular realization of the estimator, $\bar{X}$
- A fixed quantity, but the result of a random process.

Estimators are random variables

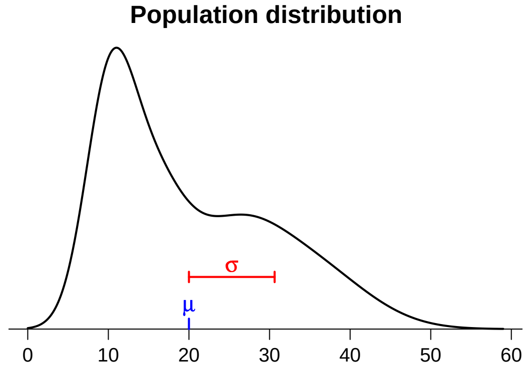
Estimators have distributions, means, SDs, etc.

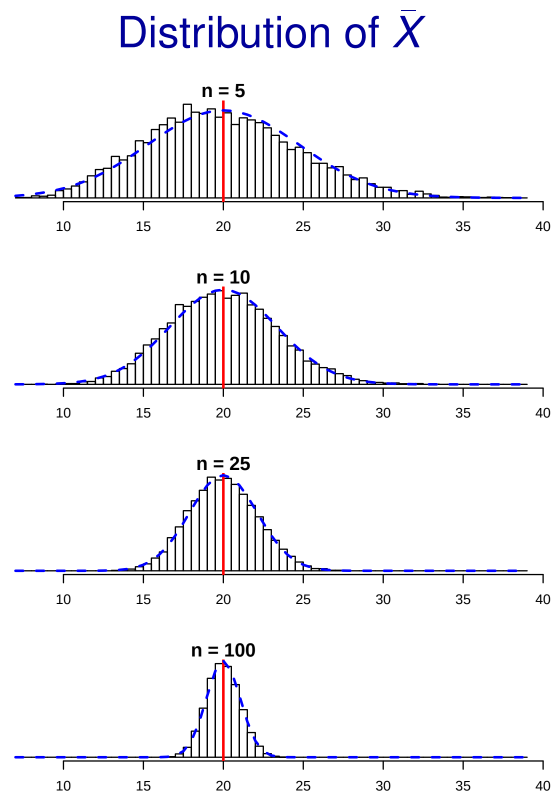


$\longrightarrow \quad X_1, X_2, \ldots, X_{10} \quad \longrightarrow \quad \bar{X}$

3.8	8.0	9.9	13.1	15.5	16.6	22.3	25.4	31.0	40.0	$\longrightarrow$	18.6
6.0	10.6	13.8	17.1	20.2	22.5	22.9	28.6	33.1	36.7	$\longrightarrow$	21.2
8.1	9.0	9.5	12.2	13.3	20.5	20.8	30.3	31.6	34.6	$\longrightarrow$	19.0
4.2	10.3	11.0	13.9	16.5	18.2	18.9	20.4	28.4	34.4	$\longrightarrow$	17.6
8.4	15.2	17.1	17.2	21.2	23.0	26.7	28.2	32.8	38.0	$\longrightarrow$	22.8

Sampling distribution

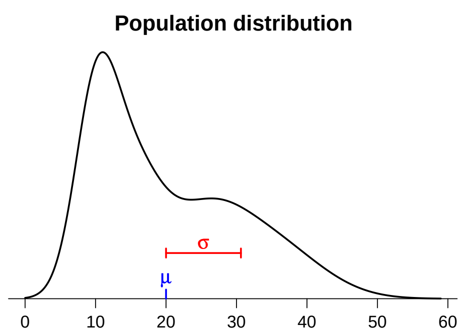


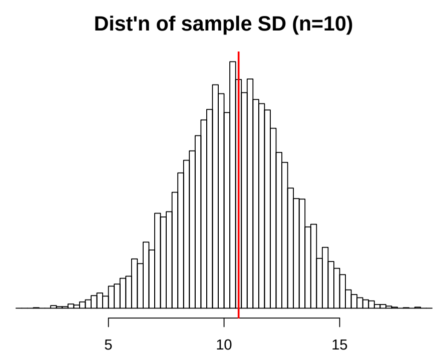


Sampling distribution depends on:

- The type of statistic
- The population distribution
- The sample size

Bias, SE, RMSE





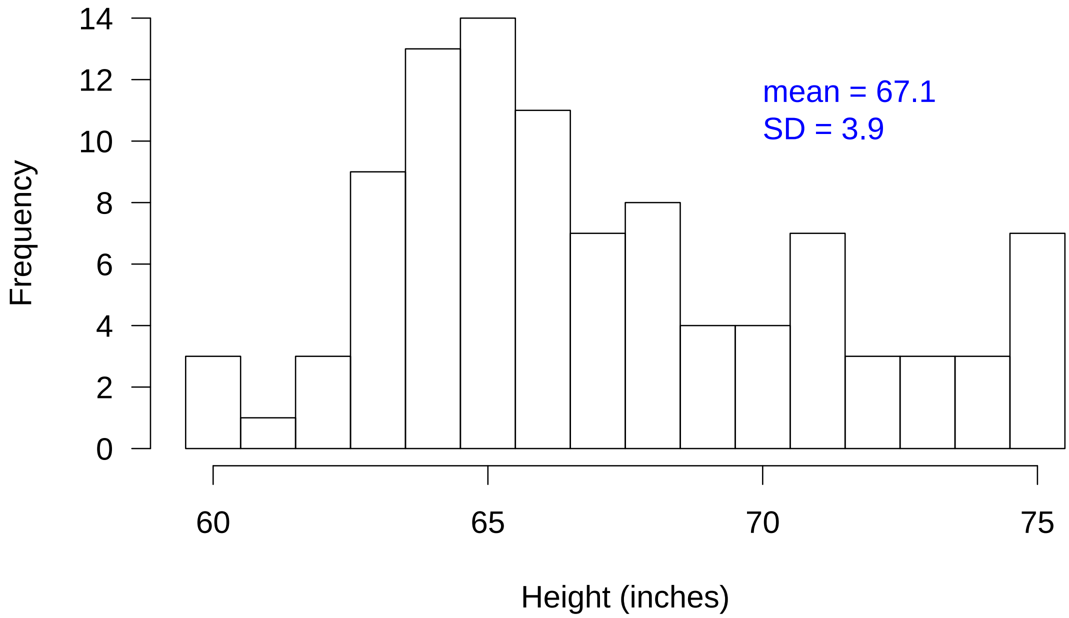
Consider $\hat{\theta}$, an estimator of the parameter θ.

Bias: $\mathrm{E}(\hat{\theta} - \theta) = \mathrm{E}(\hat{\theta}) - \theta$

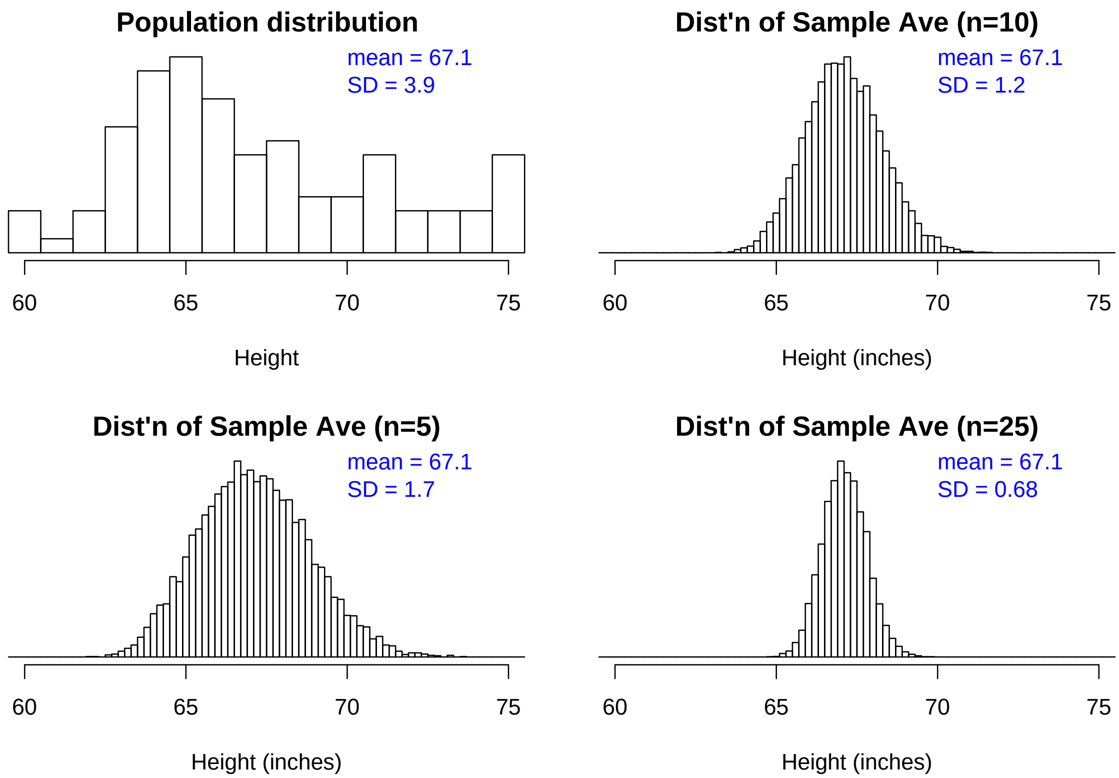
Standard error (SE): $\mathrm{SE}(\hat{\theta}) = \mathrm{SD}(\hat{\theta})$.

RMS error (RMSE): $\sqrt{\mathrm{E}\{(\hat{\theta} - \theta)^2\}} = \sqrt{(\text{bias})^2 + (\mathrm{SE})^2}$.

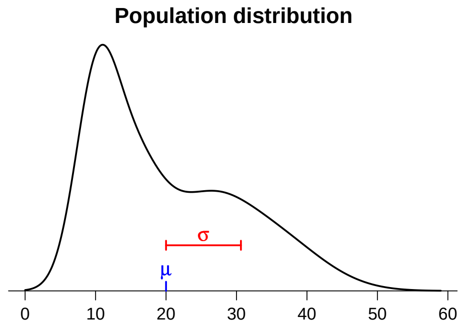
Example: heights of students



Example: heights of students



The sample mean



Assume $X_1, X_2, \ldots, X_n$ are iid with mean μ and SD σ.

Mean of $\bar{X}$ = E($\bar{X}$) = μ.

Bias = E$(\bar{X}) - \mu$ = 0.

SE of $\bar{X}$ = SD($\bar{X}$) = $\sigma/\sqrt{n}$.

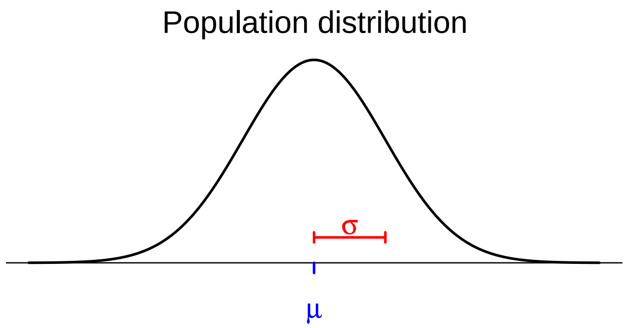
RMS error of $\bar{X}$ =

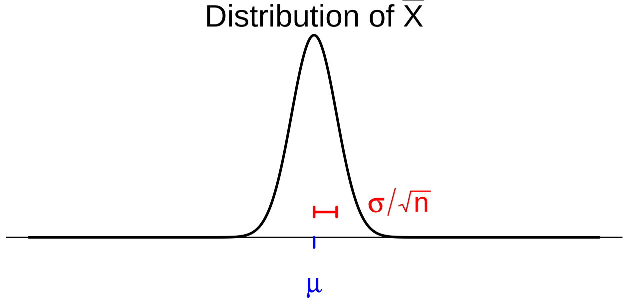
$$\sqrt{(\text{bias})^2 + (\text{SE})^2} = \sigma/\sqrt{n}.$$

If the population is normally distributed

If $X_1, X_2, \ldots, X_n$ are iid normal(μ,σ), then

$$\bar{X} \sim \text{normal}(\mu, \sigma/\sqrt{n}).$$



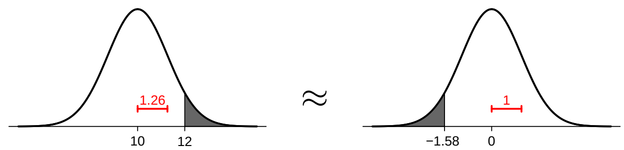


Example

Suppose $X_1, X_2, \ldots, X_{10}$ are iid normal(mean=10,SD=4)

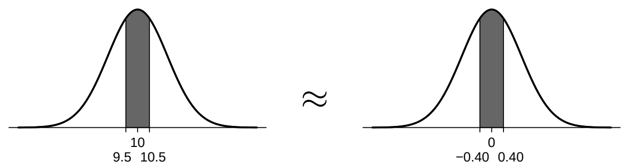
Then $\bar{X} \sim$ normal(mean=10, SD $\approx$ 1.26); let $Z = (\bar{X} - 10)/1.26$.

$\Pr(\bar{X} > 12)$?



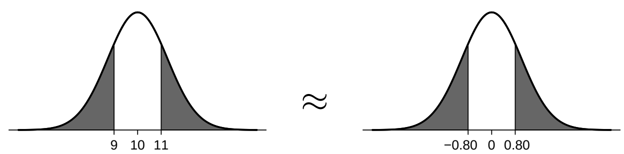
$\approx$ 5.7%

$\Pr(9.5 < \bar{X} < 10.5)$?



$\approx$ 31%

$\Pr(|\bar{X} - 10| > 1)$?



$\approx$ 43%

Central limit theorm

If $X_1, X_2, \ldots, X_n$ are iid with mean μ and SD σ.

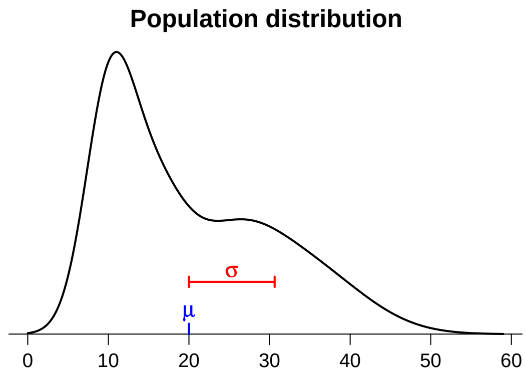
and the sample size (n) is large,

then $\bar{X}$ is approximately normal(μ, $\sigma/\sqrt{n}$).

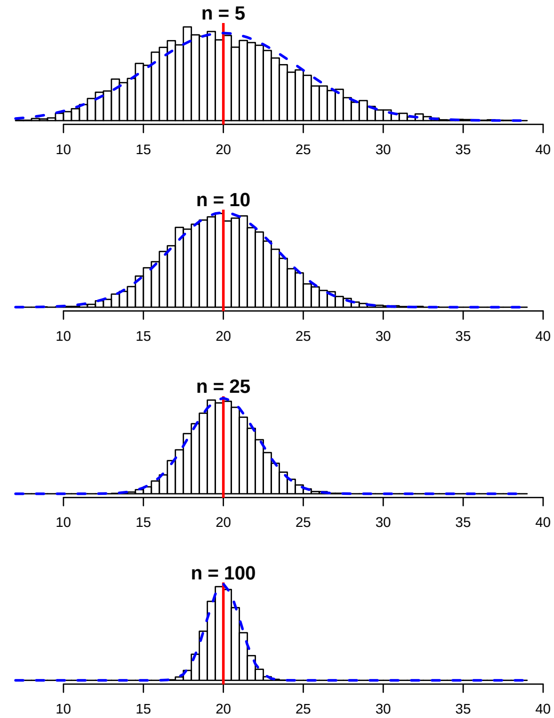
How large is large?

It depends on the population distribution.
(But, generally, not too large.)

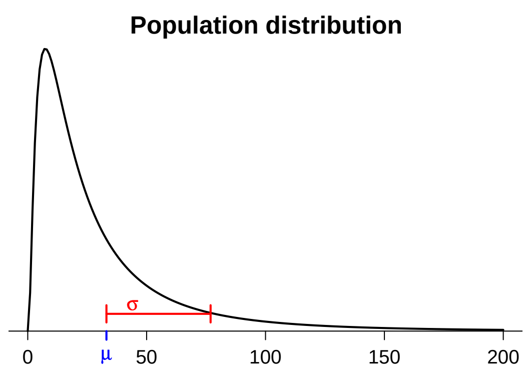
Example 1



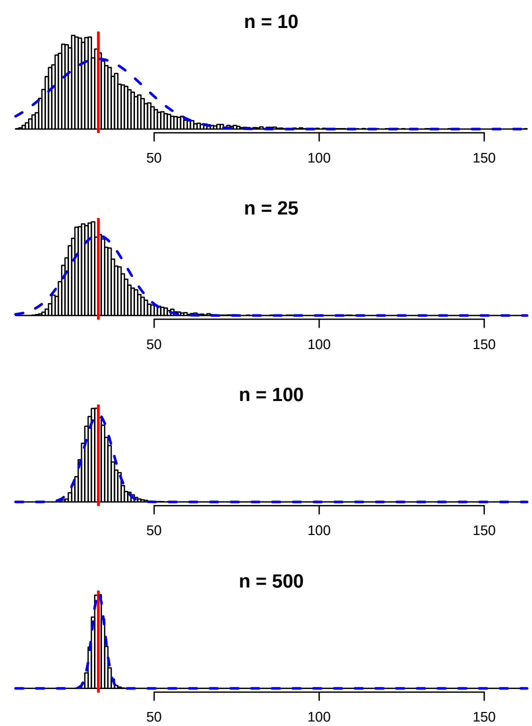
Distribution of $\bar{X}$



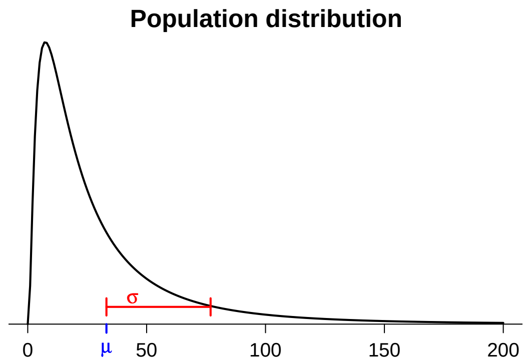
Example 2



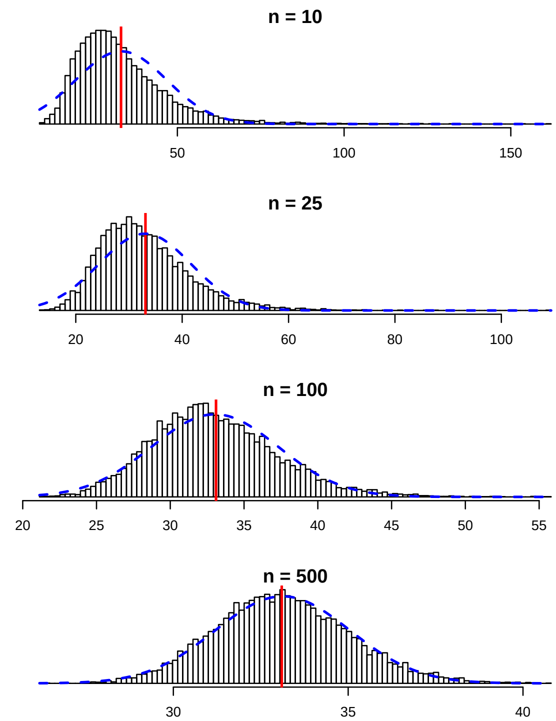
Distribution of $\bar{X}$



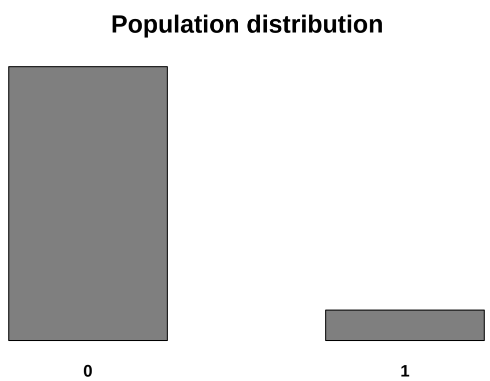
Example 2 (rescaled)



Distribution of $\bar{X}$



Example 3



$\{X_i\}$ iid

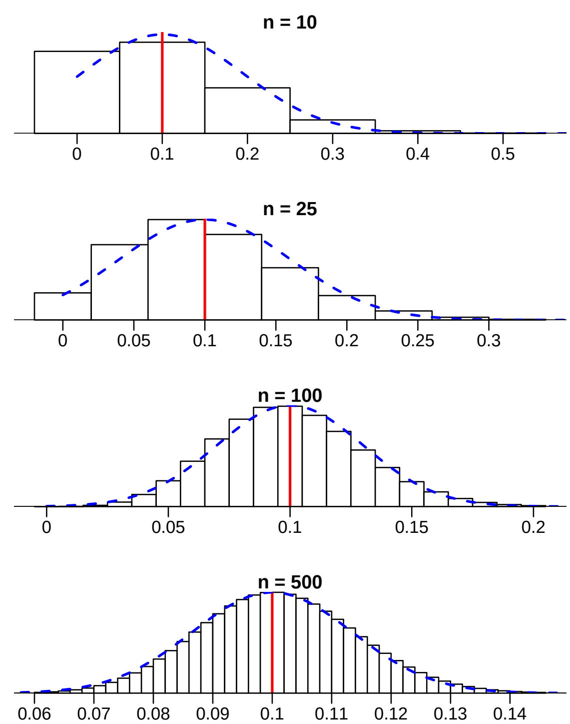
$\Pr(X_i = 0) = 90\%$
$\Pr(X_i = 1) = 10\%$

$E(X_i) = 0.1$; $SD(X_i) = 0.3$

$\sum X_i \sim \text{binomial}(n, p)$

$\bar{X}$ = proportion of 1's

Distribution of $\bar{X}$



The sample SD

Why use $(n-1)$ in the sample SD?

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

If $\{X_i\}$ are iid with mean μ and SD σ, then

$E(s^2) = \sigma^2$

but $E\{ \frac{n-1}{n} s^2 \} = \frac{n-1}{n}\sigma^2 < \sigma^2$

In other words:

$\text{Bias}(s^2) = 0$

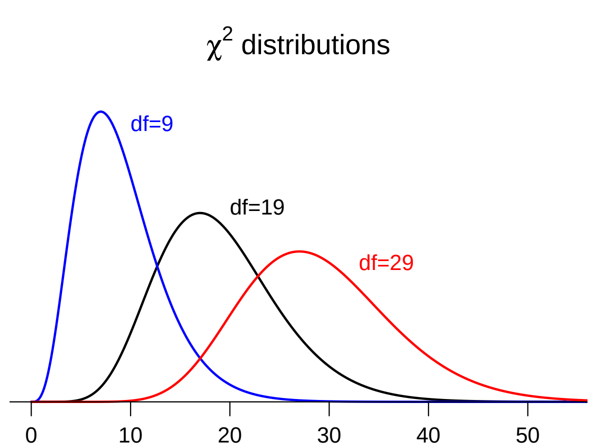
but $\text{Bias}(\frac{n-1}{n} s^2) = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2$

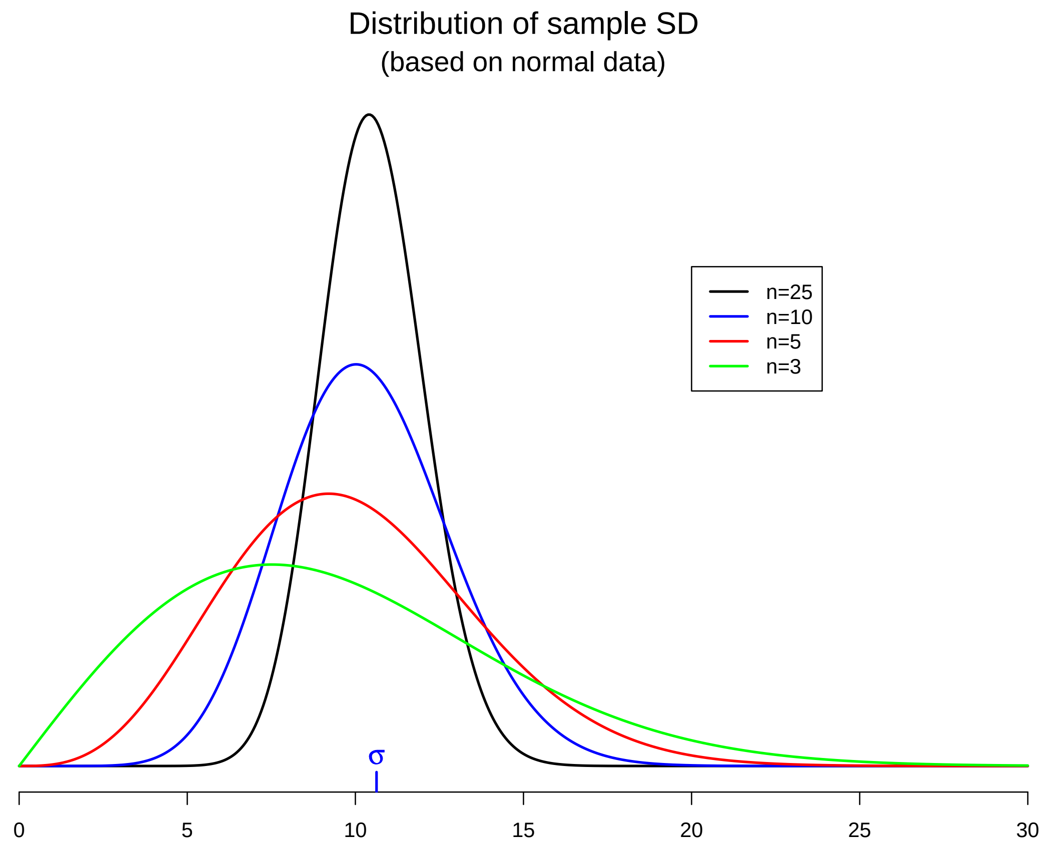
The distribution of the sample SD

If $X_1, X_2, \ldots, X_n$ are iid normal(μ, σ)

then the sample SD, s, satisfies $(n-1)\, s^2/\sigma^2 \sim \chi^2_{n-1}$

When the X_i are not normally distributed, this is not true.





A non-normal example