Statistical tests

- Gather data to assess some hypothesis (e.g., does this treatment have an effect on this outcome?)
- Form a test statistic for which large values indicate a departure from the hypothesis.
- Compare the observed value of the statistic to its distribution under the null hypothesis.

Paired t-test

Pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ independent

$X_i \sim \text{normal}(\mu_A, \sigma_A)$ $\qquad$ $Y_i \sim \text{normal}(\mu_B, \sigma_B)$

Test $H_0 : \mu_A = \mu_B$ vs $H_a : \mu_A \neq \mu_B$

Paired t-test

$D_i = Y_i - X_i$

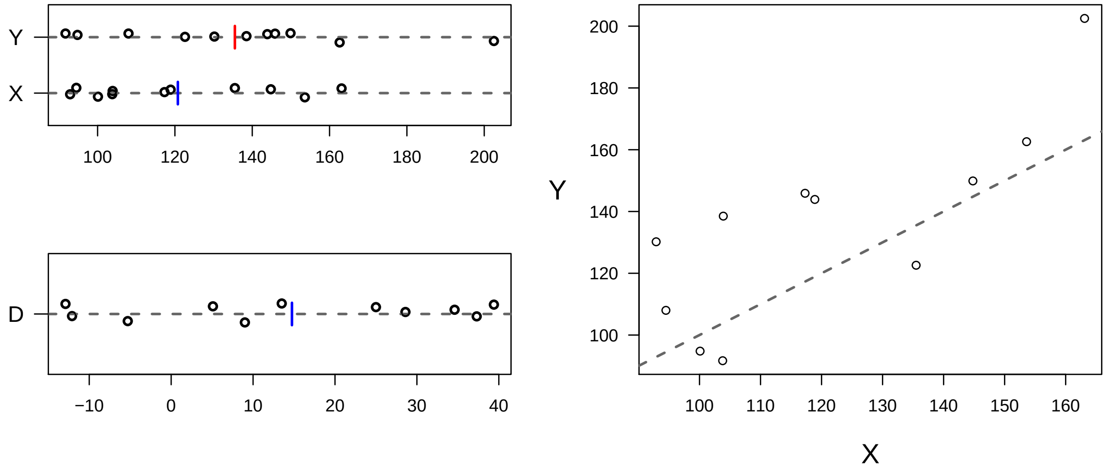
$D_1, \ldots, D_n \sim$ iid normal$(\mu_B - \mu_A, \sigma_D)$

sample mean $\bar{D}$; sample SD s_D

$T = \bar{D}/(s_D/\sqrt{n})$

Compare to t distribution with n – 1 d.f.

Example



$\bar{D} = 14.7$ $s_D = 19.6$ $n = 11$

T = 2.50 P = `2*(1-pt(2.50,10))` = 0.031

Assumptions

- Random sample from the target populations
  - Hard to check
  - Need a well-designed study
- Underlying population follows a normal distribution
  - Not necessary if the sample size is large (but large is relative)
  - Checkable, but really only if the sample size is large

Assessing normality

To assess the assumption that the underlying population follows a normal distribution, we often use a QQ plot.

- For a sample size n, look at n values evenly distributed between 0 and 1:

$$\frac{0.5}{n} \qquad \frac{1.5}{n} \qquad \frac{2.5}{n} \qquad \cdots \qquad \frac{n-0.5}{n}$$

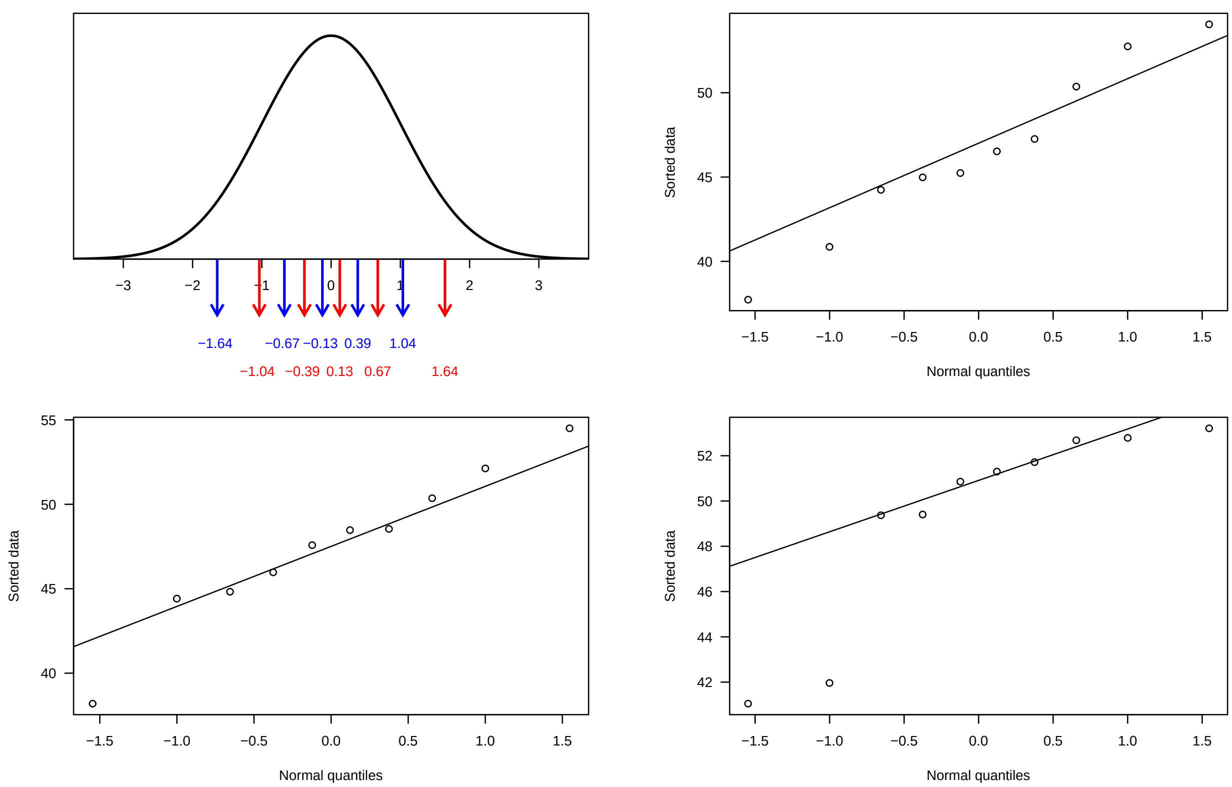
- Look at the corresponding quantiles of the normal distribution.

`qnorm(0.5/n)`   `qnorm(1.5/n)`   `qnorm(2.5/n)` $\cdots$ `qnorm((n-0.5)/n)`

i.e., `qnorm( ((1:n)-0.5)/n )`

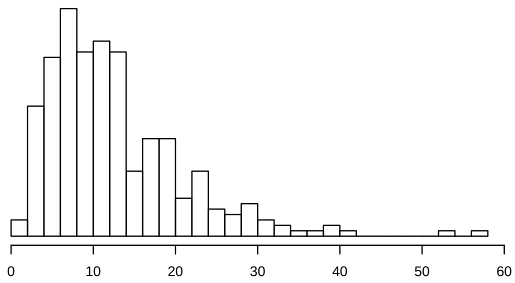
- Plot the sorted data values against these "idealized" draws from a normal distribution.
- Look for a straight line.

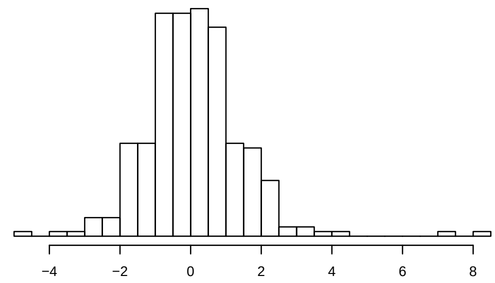
QQ plots

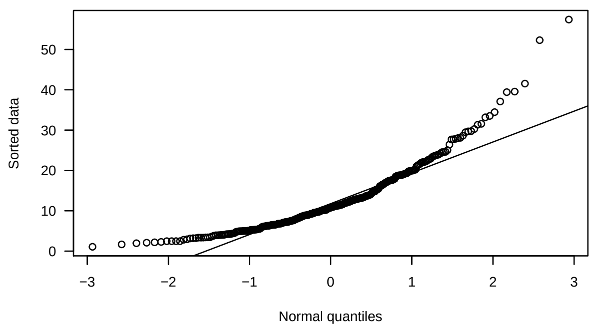


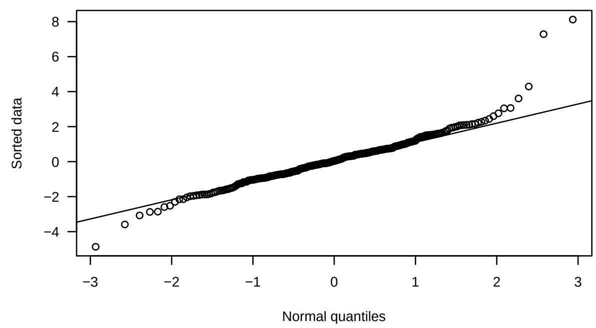
Examples

Skewed distribution | **Heavy tails**









Sign test

Suppose we are concerned about the normal assumption.

$(X_1, Y_1), \ldots, (X_n, Y_n)$ independent

Test H_0 : X's and Y's have the same distribution

Another statistic: S = $\#\{i : X_i < Y_i\} = \#\{i : D_i > 0\}$

(the number of pairs for which $X_i < Y_i$)

Under H_0, S $\sim$ binomial(n, p=0.5)

Suppose $S_{obs} > n/2$.

P-value = 2 $\times$ Pr(S $\geq S_{obs} \mid H_0$)

```
= 2 * (1 - pbinom(Sobs - 1, n, 0.5))
```

Example

For our example, 8 out of 11 pairs had $Y_i > X_i$.

P-value = `2*(1 - pbinom(7, 11, 0.5))` = 23%

Or type `binom.test(8, 11, 0.5)`.

(Compare this to P = 3% for the t-test.)

Signed Rank test

Another "nonparametric" test. (Also called the Wilcoxon signed rank test)

Rank the differences according to their absolute values.

R = sum of ranks of positive (or negative) values

D	28.6	–5.3	13.5	–12.9	37.3	25.0	5.1	34.6	–12.1	9.0	39.4
rank	8	2	6	5	10	7	1	9	4	3	11

R = 2 + 4 + 5 = 11

Compare this to the distribution of R when each rank has an equal chance of being positive or negative.

In R: `wilcox.test(d)` $\longrightarrow$ P = 0.054

Permutation test

$(X_1, Y_1), \ldots, (X_n, Y_n) \longrightarrow T_{obs}$

- Randomly flip the pairs. (For each pair, toss a fair coin. If heads, switch X and Y; if tails, do not switch.)
- Compare the observed T statistic to the distribution of the T-statistic when the pairs are flipped at random.
- If the observed statistic is extreme relative to this permutation/randomization distribution, then reject the null hypothesis (that the X's and Y's have the same distribution).

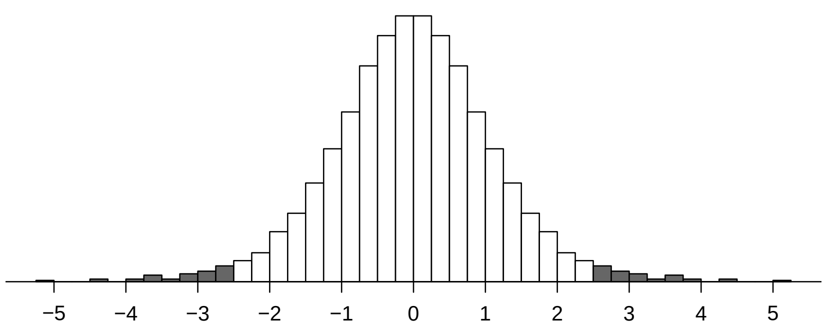
Actual data:

(117.3,145.9) (100.1,94.8) (94.5,108.0) (135.5,122.6) (92.9,130.2) (118.9,143.9)
(144.8,149.9) (103.9,138.5) (103.8,91.7) (153.6,162.6) (163.1,202.5) $\longrightarrow T_{obs} = 2.50$

Example shuffled data:

(117.3,145.9) (94.8,100.1) (108.0,94.5) (135.5,122.6) (130.2,92.9) (118.9,143.9)
(144.8,149.9) (138.5,103.9) (103.8,91.7) (162.6,153.6) (163.1,202.5) $\longrightarrow T^{\star} = 0.19$

Permutation distribution



P-value = $\Pr(|T^{\star}| \geq |T_{obs}|)$

Small n: Look at all 2^n possible flips

Large n: Look at a sample (w/ repl) of 1000 such flips

Example data:

All 2^{11} permutations: P = 0.037; sample of 1000: P = 0.040

Paired comparisons

At least four choices:

- Paired t-test
- Sign test
- Signed rank test
- Permutation test with the t-statistic

Which to use?:

- Paired t-test depends on the normality assumption
- Sign test is pretty weak
- Signed rank test ignores some information
- Permutation test is recommended

The fact that the permutation distribution of the t-statistic is generally well-approximated by a t distribution recommends the ordinary t-test. But if you can estimate the permutation distribution, do it.

2-sample t-test

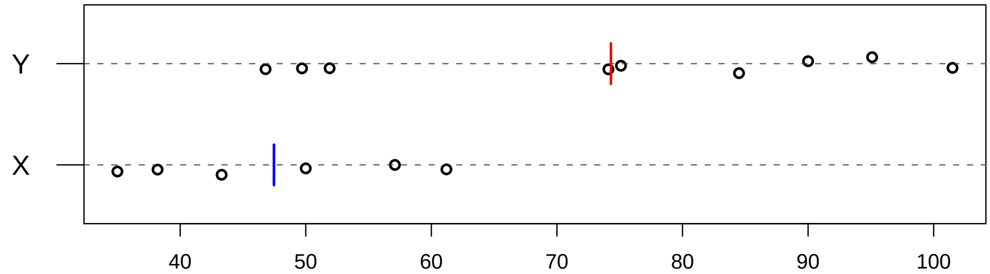
$X_1, \ldots, X_n \sim$ iid normal(μ_A, σ) $Y_1, \ldots, Y_m \sim$ iid normal(μ_B, σ)

Test $H_0 : \mu_A = \mu_B$ vs $H_a : \mu_A \neq \mu_B$

Test statistic: $T = \dfrac{\bar{X} - \bar{Y}}{s_p \sqrt{\frac{1}{n} + \frac{1}{m}}}$ where $s_p = \sqrt{\frac{s_A^2(n-1)+s_B^2(m-1)}{n+m-2}}$

Compare to t distribution with n + m – 2 degrees of freedom.

Example



$\bar{X}$ = 47.5 s_A = 10.5   n = 6

$\bar{Y}$ = 74.3 s_B = 20.6   m = 9

s_p = 17.4   T = –2.93

P = `2*pt(-2.93, 6+9-2)` = 0.011

Wilcoxon rank-sum test

Rank the X's and Y's from smallest to largest (1, 2, . . . , n+m)

R = sum of ranks for X's

(Also known as the Mann-Whitney Test)

X	Y	rank
35.0		1
38.2		2
43.3		3
	46.8	4
	49.7	5
50.0		6
	51.9	7
57.1		8
61.2		9
	74.1	10
	75.1	11
	84.5	12
	90.0	13
	95.1	14
	101.5	15

R = 1 + 2 + 3 + 6 + 8 + 9 = 29

P-value = 0.026

(use `wilcox.test()`)

Note: The distribution of R (given that X's and Y's have the same dist'n) is calculated numerically

Permutation test

X or Y	group
X_1	1
X_2	1
⋮	1
X_n	1
Y_1	2
Y_2	2
⋮	2
Y_m	2

$\longrightarrow T_{obs}$

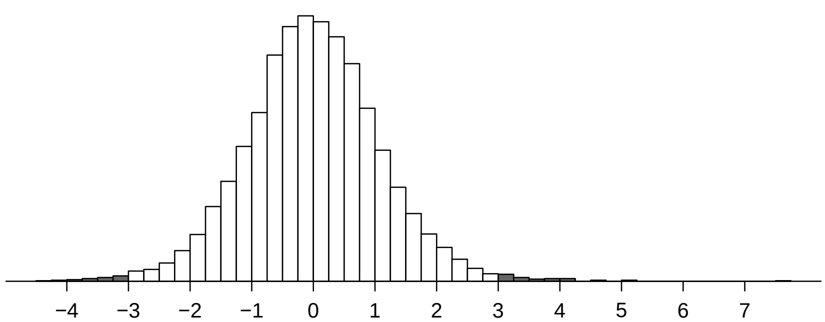
X or Y	group
X_1	2
X_2	2
⋮	1
X_n	2
Y_1	1
Y_2	2
⋮	1
Y_m	1

$\longrightarrow T^\star$

Group status shuffled

Compare the observed t-statistic to the distribution obtained by randomly shuffling the group status of the measurements.

Permutation distribution



P-value = $\Pr(|T^\star| \geq |T_{obs}|)$

Small n & m: Look at all $\binom{n+m}{n}$ possible shuffles

Large n & m: Look at a sample (w/ repl) of 1000 such shuffles

Example data:

All 5005 permutations: P = 0.015; sample of 1000: P = 0.013

Estimating the permutation P-value

Let P = true P-value (if we do *all* possible shuffles)

Do N shuffles, and let X = # times the statistic after shuffling $\geq$ the observed statistic

$\hat{P} = \frac{X}{N}$ where $X \sim$ binomial(N, P)

$E(\hat{P}) = P$ $SD(\hat{P}) = \sqrt{\frac{P(1-P)}{N}}$

If the "true" P-value P = 5% and we do N=1000 shuffles, SD($\hat{P}$)

= 0.7%.

Summary

The t-test relies on a normality assumption

If this is a worry, consider:

- Paired data:
  - Sign test
  - Signed rank test
  - Permutation test
- Unpaired data:
  - Rank-sum test
  - Permutation test

Crucial assumption: independence

The fact that the permutation distribution of the t-statistic is often closely approximated by a t distribution is good support for just doing t-tests.