QTL mapping: other single-QTL models

1. If L is measured in cM, then $L \sim \text{gamma}(\text{shape} = 2, \text{rate} = n/100)$.

   (a) $\text{E}(L) = 200/n$, and so $n \geq 200$

   (b) I used the following R code to determine that $\Pr(L \leq 1 \text{ cM}) \geq 80\%$ is true provided that $n \geq 300$.

```
f <- function(n) pgamma(1, 2, n/100) - 0.8
uniroot(f, c(1,10000))$root   # (the root is ~299.43)
```

2. We take the $\{p_{ig}\}$ as fixed and assume that the ranks $\{R_i\}$ are a random permutation of the numbers $\{1, 2, \ldots, n\}$, independent of the $\{p_{ig}\}$. We seek the mean and variance of $\bar{S}_g = \sum_i R_i p_{ig} / \sum_i p_{ig}$.

   (a) $R_i \sim \text{uniform}\{1, 2, \ldots, n\}$, so $\text{E}(R_i) = \sum_{j=1}^n j \cdot \frac{1}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$.
   Thus $\text{E}(\bar{S}_g) = \sum_i p_{ig} \text{E}(R_i) / \sum_i p_{ig} = \cdots = \frac{n+1}{2}$.

   (b) $\text{E}(R_i^2) = \sum_{j=1}^n j^2 \cdot \frac{1}{n} = \frac{(n+1)(2n+1)}{6}$, and so $\text{var}(R_i) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \cdots = \frac{(n+1)(n-1)}{12}$.
   For $i \neq j$, $\text{E}(R_i R_j) = \sum_i \sum_{j:j\neq i} \frac{i \cdot j}{n(n-1)} = \cdots = \frac{(n+1)(3n+2)}{12}$. Thus, $\text{cov}(R_i, R_j) = \text{E}(R_i R_j) - (\text{E}R_i)^2 = \frac{(n+1)(3n+2)}{12} - (\frac{n+1}{2})^2 = \cdots = -\frac{n+1}{12}$.
   Thus

$$
\begin{aligned}
\text{var}\{\sum_i R_i p_{ig} / \sum_i p_{ig}\} &= \frac{\sum_i p_{ig}^2 \text{var}(R_i) + \sum_i \sum_{j:j\neq i} p_{ig} p_{jg} \text{cov}(R_i, R_j)}{(\sum_i p_{ig})^2} \\
&= \frac{\frac{(n+1)(n-1)}{12} \sum_i p_{ig}^2 - \frac{n+1}{12} \sum_i \sum_{j:j\neq i} p_{ig} p_{jg}}{(\sum_i p_{ig})^2} \\
&= \left(\frac{n+1}{12}\right) \left\{ \frac{(n-1)\sum_i p_{ig}^2 - \sum_i \sum_{j:j\neq i} p_{ig} p_{jg}}{(\sum_i p_{ig})^2} \right\} \\
&= \left(\frac{n+1}{12}\right) \left\{ \frac{n \sum_i p_{ig}^2 - (\sum_i p_{ig})^2}{(\sum_i p_{ig})^2} \right\}
\end{aligned}
$$

   (c) Finally,

$$
H = \sum_g \left(\frac{n - \sum_i p_{ig}}{n} \right) \frac{[\bar{S}_g - \text{E}(\bar{S}_g)]^2}{\text{var}(\bar{S}_g)}
$$

   We just plug in the expressions from (a) and (b), and we are done.