Small Sample Properties of JR Estimators

John Kloke\textsuperscript{1} \quad Joseph McKean\textsuperscript{2}

\textsuperscript{1}University of Wisconsin - Madison

\textsuperscript{2}Western Michigan University

8 August 2013
Outline

- Joint ranking (JR) estimation for cluster correlated data
- Estimates of standard error
- Simulation results
- jrfit
Consider an experiment done over \( m \) blocks or clusters. For block \( k (k = 1, \ldots, m) \) we have the linear model

\[
y_k = \alpha 1_{n_k} + X_k \beta + e_k
\]

- Assume \( e_1, \ldots, e_m \) are independent random vectors.
- \( N = \sum_{k=1}^{m} n_k \).

**Matrix Formulation**

\[
y = \alpha 1_N + X \beta + e
\]

where \( X = [X_1^T \cdots X_m^T]^T \) is an \( N \times p \) design matrix and \( y = [y_1^T \cdots y_m^T]^T \) is an \( N \times 1 \) response vector.
The rank-based estimator of $\beta$ is defined as

$$
\hat{\beta}_\varphi = \text{Argmin} \| y - X\beta \| \varphi
$$

$$
= \text{Argmin} \sum_{i=1}^{N} a[R(y_i - x_i^T \beta)](y_i - x_i^T \beta)
$$

- $\| v \| \varphi = \sum_{i=1}^{N} a[R(v_i)]v_i$ for $v \in \mathbb{R}^N$, is Jaeckel’s (1972) dispersion function.
- $R$ denotes Rank.
- Scores are generated as $a[t] = \varphi \left[ \frac{t}{N+1} \right]$.
- Score function: $\varphi(u)$ is a nondecreasing, square-integrable function defined on $(0,1)$ such that $\int \varphi(u) \, du = 0$ and $\int \varphi^2(u) \, du = 1$. 
Example Score Functions

Wilcoxon (linear) scores: \( \varphi(u) = \sqrt{12} \left( u - \frac{1}{2} \right) \)

Sign scores (L1): \( \varphi(u) = \text{sign} \left( u - \frac{1}{2} \right) \)

Normal scores: \( \varphi(u) = \Phi^{-1}(u) \)

Optimal scores:
\[
\varphi(u) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}
\]
Assume equal marginals:

\[ e_{kj} \sim f, F \text{ for } k = 1, \ldots, m, j = 1, \ldots n_k \]

The joint rankings (JR) estimator of \( \beta \) is

\[ \hat{\beta}_{JR} = \text{Argmin} \| y - X\beta \|_\varphi, \]

where

\[ \| v \|_\varphi = \sum_{t=1}^{N} a(R(v_t))v_t \]

is Jaeckel’s (1972) dispersion function.
Asymptotic Distribution
Kloke, McKean, Rashid (2009)

Using a result from Brunner and Denker (1994)

\[ \hat{\beta}_{JR} \sim N_p \left( \beta, \tau^2 \phi \left( \mathbf{X}^T \mathbf{X} \right)^{-1} \mathbf{V}_\phi \left( \mathbf{X}^T \mathbf{X} \right)^{-1} \right) \]

- \[ \mathbf{V}_\phi = \sum_{k=1}^m \mathbf{X}^T_k \Sigma_{\varphi,k} \mathbf{X}_k \]
- \[ \Sigma_{\varphi,k} = \text{var}(\varphi(F(e_k))) \]
- \[ \varphi(F(e_k)) = [\varphi(F(e_{k1})), \ldots, \varphi(F(e_{kn_k}))]^T \]
- \( \tau_\phi \) is a scale parameter which can be estimated using the method proposed by Koul, Sievers, McKean (1987).
Estimates of $\mathbf{V}_\varphi = \sum_{k=1}^{m} \mathbf{X}_k^T \Sigma_{\varphi} \mathbf{X}_k$

1. Assume $\Sigma_{\varphi}$ is **compound symmetric**. Then

$$\hat{\Sigma}_{\varphi} = (1 - \hat{\varphi}) \mathbf{I} + \hat{\varphi} \mathbf{J}$$

where $\hat{\varphi} = \frac{1}{M-p} \sum_{k=1}^{m} \sum_{i>j} \text{a}(R(\hat{e}_{ki})) \text{a}(R(\hat{e}_{kj}))$, $M = \sum_{k=1}^{m} \binom{n_k}{2}$.

2. **Sandwich** estimator.

$$\frac{m}{m-p} \sum_{k=1}^{m} \mathbf{X}_k^T \text{a}(R(\hat{e}_k)) \text{a}(R(\hat{e}_k))^T \mathbf{X}_k$$
Simulation 1

\[ y_{kj} = \alpha + x_{kj}^T \beta + w_{kj}^T \Delta + b_k + e_{kj} \]

\[ \rho = 0.75 \]

\[ e, b, x \sim \mathcal{N} \]

\[ k = 2 \text{ (number of treatments)} \]

\[ p = 1 \text{ (number of covariates)} \]

\[ m = 4, 8, 16, 32 \text{ (number of blocks)} \]

\[ n = 4 \text{ (block size)} \]

\[ \text{treatments are balanced within blocks} \]
Empirical Level

H₀: β₁ = 0

m

H0:β₁= 0

empirical level

m

Kloke, McKean
Small Sample Properties of JR Estimators
Empirical Level

H₀: Δ₁ = 0

0.00 0.02 0.04 0.06 0.08 0.10

m

empirical level

5 10 15 20 25 30

Kloke, McKean

Small Sample Properties of JR Estimators
Simulation 2

\[ y_{kj} = \alpha + x_{kj}^T \beta + w_{kj}^T \Delta + b_k + e_{kj} \]

\[ \rho = 0.1, 0.25, 0.75, 0.9 \]

\[ e, b, x \sim N \]

\[ k = 3 \text{ (number of treatments)} \]
\[ p = 1 \text{ (number of covariates)} \]
\[ m = 4, 8, 16, 32 \text{ (number of blocks)} \]
\[ n = 6 \text{ (block size)} \]

treatments are balanced within blocks
Empirical Level: $H_0 : \Delta_i = 0$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Empirical Level</th>
<th>m</th>
<th>$\rho$</th>
<th>Empirical Level</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.00 0.04 0.08</td>
<td></td>
<td>0.25</td>
<td>0.00 0.04 0.08</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.00 0.04 0.08</td>
<td></td>
<td>0.9</td>
<td>0.00 0.04 0.08</td>
<td></td>
</tr>
</tbody>
</table>

Kloke, McKean  
Small Sample Properties of JR Estimators
Simulation 3

\[ y_{kj} = \alpha + x_{kj}^T \beta + w_{kj}^T \Delta + b_k + e_{kj} \]
\[ \rho = 0.1, 0.9 \]
\[ e, b, x \sim N \]
\[ k = 2, 3 \text{ (number of treatments)} \]
\[ p = 1, 3 \text{ (number of covariates)} \]
\[ m = 32 \text{ (number of blocks)} \]
\[ n = 6, 12, 18, 24 \text{ (block size) i.e. 8 of each} \]
\[ \text{treatments are balanced within blocks} \]
Empirical Level: $H_0 : \Delta_i = 0 (\rho = 0.1)$

Kloke, McKean  
Small Sample Properties of JR Estimators
Empirical Level: $H_0 : \Delta_i = 0 (\rho = 0.9)$

Kloke, McKean  
Small Sample Properties of JR Estimators
Simulation 4 (Repeated Measures)

\[ y_{kt} = \Delta_t w_k + b_k + e_{kt} \text{ for } k = 1, \ldots, m; t = 1, \ldots, \max T. \]

- \( \max T = 4, 8 \).
- \( e, b \sim N \)
- \( m = 12, 25, 50, 75, 100, 150, 200 \).
- \( k = 2 \) treatments (assigned at random)
- test of parallel profiles (k1) and test of equal mean (k2)
Empirical Level: $H_0 : K_1 \Delta = 0$ (maxT = 4)

Kloke, McKean  
Small Sample Properties of JR Estimators
Empirical Level: $H_0: K_1 \Delta = 0 \ (\text{max}T = 8)$

- $\text{max}T= 8 \ \rho = 0.1$
- $\text{max}T= 8 \ \rho = 0.25$
- $\text{max}T= 8 \ \rho = 0.75$
- $\text{max}T= 8 \ \rho = 0.9$

Kloke, McKean  
Small Sample Properties of JR Estimators
Empirical Level: $H_0 : K_2\Delta = 0$ (maxT = 4)

maxT= 4 rho= 0.1

maxT= 4 rho= 0.25

maxT= 4 rho= 0.75

maxT= 4 rho= 0.9

Kloke, McKea
Small Sample Properties of JR Estimators
Empirical Level: $H_0 : K_2 \Delta = 0$ (maxT = 8)

Kloke, McKean
Small Sample Properties of JR Estimators
> args(jrfit)
function (x, y, block, yhat0 = NULL, scores = wscores, fitint = NULL, var.type = "sandwich", fitblock = FALSE, ...)

- **x**: $N \times p$ design matrix
- **y**: $N \times 1$ vector of responses
- **block**: $N \times 1$ vector denoting block membership
- **var.type**: ‘sandwich’, ‘cs’, ‘ind’, or ‘user’.
- **fitblock**: should block be fit as fixed effect?
Example: Simulated Cluster-Correlated

- Model: \( y_{ij} = \alpha + w_i \Delta + x_{ij} \beta + b_j + e_{ij} \)
- \( m = 160 \) blocks
- \( n = 4 \) observations (repeated measures) per block
- \( \Delta = 0.5 \) is the treatment effect
- \( \beta = 0 \) is the effect of a (baseline) covariate
- \( e_{ij} \sim t_5 \) and \( b_j \sim t_3 \)
```r
> fit<-jrfit(cbind(x,w),y,block)
> summary(fit)

Coefficients:
   Estimate  Std. Error  t-ratio  p.value
x    0.090489  0.139048   0.6508  0.516126
w    0.709443  0.257816   2.7517  0.006612

> fit<-jrfit(cbind(x,w),y,block, var.type='cs')
> summary(fit)

Coefficients:
    Estimate   Std. Error  t-ratio  p.value
x     0.090489  0.118293   0.7649  0.445426
w    0.709443  0.264278   2.6845  0.008029
```

Kloke, McKeane
Small Sample Properties of JR Estimators
Sandwich estimator works well in all cases considered.

jrfit provides robust rank-based inference for linear model w/ cluster correlated errors. Available at http://www.biostat.wisc.edu/~kloke/.
Future Work: Generalized JR

- Obtain a consistent, robust estimate of $\text{var}(y_k) := \hat{V}_k$
- Transform responses to working independence
  $$y_k^* = \hat{V}^{-1/2} y_k$$
- Apply JR estimation.
References


